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Problems Quantum Field Theory

Sheet 4

Problem 14: Charge conjugation for the complex scalar field

The charge conjugation operation for the quantized complex scalar field is defined by

$$\phi'(x) = \mathcal{C}\phi(x)\mathcal{C}^\dagger = \eta_\mathcal{C}\phi^\dagger(x) \quad ,$$

where the charge conjugation operator $\mathcal{C}$ is unitary and leaves the vacuum state invariant, $\mathcal{C}|0\rangle = |0\rangle$. $\eta_\mathcal{C}$ is a complex number.

1. Show that, for the annihilation operators $a_\mathbf{p}$ and $b_\mathbf{p}$,

$$\mathcal{C}a_\mathbf{p}\mathcal{C}^\dagger = \eta_\mathcal{C} b_\mathbf{p} \quad , \quad \mathcal{C}b_\mathbf{p}\mathcal{C}^\dagger = \eta_\mathcal{C}^* a_\mathbf{p}$$

2. Consider the one-particle states

$$|a,\mathbf{p}\rangle = a_\mathbf{p}^\dagger|0\rangle \quad , \quad |b,\mathbf{p}\rangle = b_\mathbf{p}^\dagger|0\rangle$$

and show their transformation behavior under charge conjugation,

$$\mathcal{C}|a,\mathbf{p}\rangle = \eta_\mathcal{C}^*|b,\mathbf{p}\rangle \quad , \quad \mathcal{C}|b,\mathbf{p}\rangle = \eta_\mathcal{C}|a,\mathbf{p}\rangle.$$

3. Show that the Lagrangian describing a free complex scalar field is invariant under charge conjugation.

4. How does the charge operator (discussed on exercise sheet 3) transform under charge conjugation?

Problem 15: Derivation of the Yukawa potential from Quantum Field Theory

Consider a real scalar field $\phi(x)$ in the presence of a classical static source described by the density $S(\mathbf{x})$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + S\phi \quad .$$

1. Find the Hamiltonian of the theory and write down the classical equations of motion.

2. The quantization of the scalar field can be summarized as follows:

$$\begin{aligned}
\pi &= \frac{\partial\mathcal{L}}{\partial\dot\phi} \,, \qquad [\pi(\mathbf{x}),\phi(\mathbf{y})] = -i\delta(\mathbf{x}-\mathbf{y}) \quad ,\\
\phi(\mathbf{x}) &= \int d\mu(\mathbf{p}) \left(a_\mathbf{p}u_\mathbf{p}(\mathbf{x}) + a_\mathbf{p}^\dagger u_\mathbf{p}^*(\mathbf{x})\right) \quad ,\\
\pi(\mathbf{x}) &= \frac{1}{i}\int d\mu(\mathbf{p})\omega_\mathbf{p} \left(a_\mathbf{p}u_\mathbf{p}(\mathbf{x}) - a_\mathbf{p}^\dagger u_\mathbf{p}^*(\mathbf{x})\right) \quad ,
\end{aligned}$$

where $u_{\mathbf{p}} = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{x}}$. Construct the Hamiltonian of the quantized theory using the Fourier components of $S(\mathbf{x})$,

$$S(\mathbf{x}) = \int d^3p \, \frac{1}{(2\pi)^{3/2}} S_{\mathbf{p}} \, e^{i\mathbf{p}\cdot\mathbf{x}} \quad .$$

Use the fact that the density $S(\mathbf{x})$ is a real quantity in order to express $S_{-\mathbf{p}}$ in terms of $S^*_{\mathbf{p}}$. Why is this the case?

3. Use the following canonical transformation,

$$a_{\mathbf{p}} = b_{\mathbf{p}} + \eta_{\mathbf{p}} \quad , \quad \text{where} \quad \eta_{\mathbf{p}} \in C \quad ,$$

to rewrite the Hamiltonian in such a way that only terms of the form $b^\dagger_{\mathbf{p}} b_{\mathbf{p}}$ and complex valued terms remain. This is possible by choosing an appropriate $\eta_{\mathbf{p}} \in C$.

How does the spectrum of the Hamiltonian change in the presence of the source as compared to the free Hamiltonian (i.e., $S = 0$)? What is the physical interpretation of this result?

4. Now calculate explicitly the interaction energy of two point charges q_1 and q_2, located at positions $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively, i.e. use

$$S(\mathbf{x}) = q_1 \delta(\mathbf{x} - \mathbf{x}_1) + q_2 \delta(\mathbf{x} - \mathbf{x}_2) \quad .$$

There will be divergent self-energy contributions appearing in the calculation. How can you identify them and distinguish them from the interaction energy of the two charges at two different spacetime points? Calculate the interaction energy. Does the result seem familiar? What is the physics behind it? What happens in the limit $m \to 0$?