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Problems Quantum Field Theory

Sheet 2

Problem 4: Anti-commutation relations

The interpretation of the operators $a_i, a_i^\dagger$ (i denotes the particle species or the quantum number of the particle) as annihilation and creation operators relies on their commutation relations with the number operators $N_i = a_i^\dagger a_i$:

$$[N_i, a_j] = -\delta_{ij} a_j \quad , \quad [N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger$$

and the existence of a vacuum state $|0\rangle$ with $a_i|0\rangle = 0$ for all i.

1. Prove the operator identity $[AB, C] = A\{B, C\} - \{A, C\}B$.

2. Consider the operators $a_i, a_i^\dagger$, which obey the anti-commutation relations

$$\{a_i, a_j^\dagger\} = \delta_{ij} \quad ,$$

while all other anti-commutators vanish, and show that these operators obey the above commutation relations with the number operators.

Problem 6: D-function

In the lecture the following D-function plays an important role. Prove the identity

$$D(t, \mathbf{x}) = -\frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} e^{i\mathbf{k}\cdot\mathbf{x}} \sin(\omega_k t) = \frac{1}{8\pi r} \left(\delta(t+r) - \delta(t-r)\right)$$

where $\omega_k = |\mathbf{k}|$.

Problem 7: Infinitesimal Lorentz transformation

An infinitesimal Lorentz transformation can be written in the form

$$\Lambda = \mathbb{1} + \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} \quad .$$

Consider the case $\omega_{12} = \frac{\theta}{N} = -\omega_{21}$ with all the other entries of ω vanishing. Show that in the limit $N \to \infty$ the N-fold application of Λ leads to a rotation about the z-axis with rotation angle θ.

Problem 8: Noether currents of Lorentz transformations

The 6 Noether currents associated with Lorentz transformations are given by

$$M^{\rho\mu\nu} = \frac{1}{2} \left(x^\mu T^{\rho\nu} - x^\nu T^{\rho\mu} \right)$$

and lead to 6 conserved Noether charges $J_{\mu\nu} = \int d^3x M_{0\mu\nu} = -J_{\nu\mu}$. Now consider a real scalar field $\phi(x)$ and

1. show that the 3 charges J_{ij} in the Hamiltonian formalism read

$$J_{ij} = \frac{1}{2} \int d^3x \, \pi(x) \left(x_i \partial_j - x_j \partial_i \right) \phi(x) \quad ,$$

2. determine the operators L_{ij} in the Poisson brackets

$$\{ J_{ij}, \phi(x) \} = L_{ij} \phi(x) \quad ,$$

3. compute the Poisson brackets of the J_{ij}.

Problem 9: Complex scalar field

Let $\phi(x)$ be a complex scalar field obeying the Klein-Gordon equation. The action of the theory is given by

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right) \quad .$$

1. Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$.

2. Compute the Heisenberg equations of motion and show that they are indeed equivalent to the Klein-Gordon equation.