Chapter 14

Path integral for gauge fields

All fundamental theories in particle physics are gauge theories. These theories contain first class constraints which generate the (time-independent) gauge transformations and hence must be quantized along the lines outlined above. We shall first recall the classical canonical structure of pure Yang-Mills theories with particular emphasis on the constraints. At the end we specialize to the Abelian case and set some of the potentials and field strengths to zero to recover the path integral for the Schwinger model.

14.1 Classical Yang-Mills Theories

In Minkowski spacetime the Lagrangian for a non-Abelian gauge theory reads

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}, \tag{14.1}$$

where the (hermitian) field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. The chromoelectric and chromomagnetic fields are the generalization of the electric and magnetic fields in electromagnetism,

$$F_{0i} = E_i \quad \text{and} \quad F_{ij} = -\epsilon_{ijk}B_k \tag{14.2}$$

Expanding the potential and field strength as

$$A^\mu = \sum_{a=1}^{\dim G} A^\mu_a T_a \quad , \quad F^{\mu\nu} = \sum_{a=1}^{\dim G} F^{\mu\nu}_a T_a,$$

where the (hermitian) generators T_a of the Lie algebra obey the commutation relations

$$[T_a, T_b] = i f_{abc} T_c, \tag{14.3}$$

with totally anti-symmetric and real structure constants f_{abc}, we find the following formulae for the components in group-space,

$$\boldsymbol{E}_a = \frac{d}{dt}\boldsymbol{A}_a - \nabla A^0_a + f_{abc}A^0_b\boldsymbol{A}_c \quad , \quad \boldsymbol{B}_a = -\nabla\times\boldsymbol{A}_a - \frac{1}{2}f_{abc}\boldsymbol{A}_b\times\boldsymbol{A}_c. \tag{14.4}$$

We have set $\boldsymbol{A} = (A_1, A_2, A_3),\ \boldsymbol{E} = (E_1, E_2, E_3)$ and $\boldsymbol{B} = (B_1, B_2, B_3)$. One would have the usual sign convention [51] if one would take $\boldsymbol{A} = (A^1, A^2, A^3)$ that is replace $\boldsymbol{A}$ by $-\boldsymbol{A}$. In the non-covariant notation the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\sum_a(\boldsymbol{E}_a^2 - \boldsymbol{B}_a^2). \tag{14.5}$$

The non-covariant form of the Yang-Mills equations $D_\nu F^{\mu\nu}$ are the generalized Gauss- and Ampere law

$$\begin{aligned} D\cdot\boldsymbol{E} = 0 \quad &\Longleftrightarrow \quad \nabla\cdot\boldsymbol{E}_a + f_{abc}\boldsymbol{A}_b\cdot\boldsymbol{E}_c = 0 \\ D_t\boldsymbol{E} = (D\times\boldsymbol{B}) \quad &\Longleftrightarrow \quad \partial_t\boldsymbol{E}_a + f_{abc}A^0_b\boldsymbol{E}_c = \nabla\times\boldsymbol{B}_a + f_{abc}(\boldsymbol{A}_b\times\boldsymbol{B}_c). \end{aligned} \tag{14.6}$$

The corresponding identities in two dimensions for $F_{01} = E$ are obtained by setting $\boldsymbol{E} = (E, 0, 0),\ \boldsymbol{B} = 0$ and $A_2 = A_3 = 0$ in the above equations.

14.1.1 Hamiltonian structure

Our task is to build a Hamiltonian scheme, which will give rise to these Yang-Mills equations. The first problem in passing to a Hamiltonian description arises from the fact that $\mathcal{L}$ does not depend on $\dot{A}^0_a$ and thus there is no momentum conjugate to A^0_a. To remedy this we use the gauge freedom to choose the temporal gauge $A^0_a = 0$. In this gauge we have

$$\mathcal{L} = \frac{1}{2}(\dot{\boldsymbol{A}}_a^2 - \boldsymbol{B}_a^2) \tag{14.7}$$

and the Gauss- and Ampere laws take the simple forms

$$(D\cdot\boldsymbol{E})_a = 0 \quad \text{and} \quad \dot{\boldsymbol{E}}_a = (D\times\boldsymbol{B})_a. \tag{14.8}$$

The momentum density conjugate to $\boldsymbol{A}_a$ is gotten by differentiating $\mathcal{L}$ in (14.7) with respect to the 'velocity' $\dot{\boldsymbol{A}}$,

$$\pi_a(x) = \frac{\delta\mathcal{L}}{\delta\dot{\boldsymbol{A}}_a(x)} = \dot{\boldsymbol{A}}_a = \boldsymbol{E}_a \tag{14.9}$$

which then lead to the following Hamiltonian and Hamiltonian density,

$$H = \int d^3x\,\mathcal{H}, \quad \text{where} \quad \mathcal{H} = \frac{1}{2}(\boldsymbol{E}_a^2 + \boldsymbol{B}_a^2). \tag{14.10}$$

The canonical equal time commutation relations read (here we do not distinguish between upper and lower indices, in particular $A^i = A_i$)

$$\{A^i_a(t,x), E^j_b(t,y)\} = \delta_{ab}\delta_{ij}\delta^3(x-y), \tag{14.11}$$

from which follows that

$$\{B^i_a(t,x), E^j_b(t,y)\} = \epsilon_{ijk}\Big(\delta_{ab}\partial_{x^k}\delta(x-y) - f_{abc}A^k_c\delta(x-y)\Big). \tag{14.12}$$

Now it is rather straightforward to calculate the time-derivative of the canonical fields. On obtains

$$\dot{A}^i_a(x) = \{A^i_a(x), H\} = \int d^3y\ \{A^i_a(x), E^j_b(y)\}E^j_b(y) = E^i_a(x) \tag{14.13}$$

and similarly, using (14.11),

$$\dot{E}^i_a(x) = \{E^i_a(x), H\} = \epsilon_{ijk}\Big(\partial_j B^k_a + f_{abc}A^j_b B^k_c\Big) \tag{14.14}$$

and hence the Hamiltonian equations reproduce Ampere's law (14.8) and the definition of $\boldsymbol{E}_a$ in terms of $\dot{\boldsymbol{A}}_a$. However, Gauss's law has yet not emerged, since it is a fixed-time constraint between canonical variables.

To understand the role of the Gauss constraints

$$C_a(x) = (D\cdot\boldsymbol{E})_a = \partial_i E^i_a + f_{abc}A^i_b E^i_c \tag{14.15}$$

more clearly let us calculate the commutator of these constraints with the canonical variables. One finds

$$\begin{aligned}\{\boldsymbol{A}_b(y), C_a(x)\} &= \delta_{ab}\nabla_x\delta(x-y) - f_{abc}\boldsymbol{A}_c\delta(x-y)\\ \{\boldsymbol{E}_b(y), C_a(x)\} &= -f_{abc}\boldsymbol{E}_c\delta(x-y).\end{aligned} \tag{14.16}$$

Smearing the constraints with arbitrary test functions θ^a as

$$C_\theta = \int d^3x\theta_a(x)C_a(x), \tag{14.17}$$

these commutation relations become

$$\begin{aligned}\{\boldsymbol{A}_a(y), C_\theta\} &= -\nabla\theta_a(y) + f_{abc}\theta_b(y)\boldsymbol{A}_c(y)\\ \{\boldsymbol{E}_a(y), C_\theta\} &= f_{abc}\theta_b(y)\boldsymbol{E}_c(y).\end{aligned} \tag{14.18}$$

From the first equation one may obtains

$$\{\boldsymbol{B}_a(y), C_\theta\} = f_{abc}\theta_b(y)\boldsymbol{B}_c(y). \tag{14.19}$$

Now we shall see, that the constraints generate the time-independent gauge transformations

$$\boldsymbol{A} \longrightarrow e^{-i\theta}\boldsymbol{A}e^{i\theta} + ie^{-i\theta}\nabla e^{i\theta}, \quad \boldsymbol{E} \longrightarrow e^{-i\theta}\boldsymbol{E}e^{i\theta}, \quad \boldsymbol{B} \longrightarrow e^{-i\theta}\boldsymbol{B}e^{i\theta}. \tag{14.20}$$

The corresponding small transformations of the gauge potential and field strengths are

$$\delta_\theta \boldsymbol{A} = -\nabla\theta - i[\theta, \boldsymbol{A}] \quad \delta_\theta \boldsymbol{E} = -i[\theta, \boldsymbol{E}] \quad \text{and} \quad \delta_\theta \boldsymbol{B} = -i[\theta, \boldsymbol{B}], \tag{14.21}$$

which, after expanding $\theta = \theta_a T_a$ read in component form

$$\delta_\theta \boldsymbol{A}_a = -\nabla\theta_a + f_{abc}\theta_b \boldsymbol{A}_c, \quad \delta_\theta \boldsymbol{E}_a = f_{abc}\theta_b \boldsymbol{E}_c \quad \text{and} \quad \delta_\theta \boldsymbol{B}_a = f_{abc}\theta_b \boldsymbol{B}_c \tag{14.22}$$

which are identical with the corresponding commutation relations in (14.18,14.19) with the smeared constraint C_θ. Hence the Gauss-constraints generate the time-independent gauge transformations.

It follows then that the Hamiltonian commutes with the constraints since it is gauge invariant. Finally, using the identity

$$f(y)\delta'(x-y) = f(x)\delta'(x-y) + f'(x)\delta(x-y) \tag{14.23}$$

and the Jacobian identity

$$f_{abd}f_{cpd} + f_{cad}f_{bpd} + f_{bcd}f_{apd} = 0 \tag{14.24}$$

one shows that the commutator of two different constraints follow the Lie algebra of the gauge group,

$$\{C_a(x), C_b(y)\} = f_{abc}C_c(x)\delta(x-y), \tag{14.25}$$

and thus form a system of first class constraints. The transition from the classical Poisson bracket to the corresponding commutators is as usual achieved by replacing Poisson brackets $\{.,.\}$ by commutators $-i[.,.]/\hbar$ in the above relations.

The path integral for the Yang-Mills Hamiltonian (14.10) is given by analogy with the constrained quantum mechanical system (13.19) by

$$Z = \int \mathcal{D}\boldsymbol{E}_a \mathcal{D}\boldsymbol{A}_a \delta(C_a)\delta(F_a)\det\{F_a, C_b\}\exp\Big[\frac{i}{\hbar}\int (\boldsymbol{E}_a\dot{\boldsymbol{A}}_a - \frac{1}{2}\boldsymbol{E}_a^2 - \frac{1}{2}\boldsymbol{B}_a^2)dtd^3x\Big], \tag{14.26}$$

where the F_a are the gauge fixing depending on $\boldsymbol{A}_a$. We have seen that $\int \theta^a C_a$ generates infinitesimal gauge transformations, and hence $\{F_a, C_b\}$ is just an infinitesimal gauge transformation with parameters θ^a stripped off

$$\{F_b(\boldsymbol{A}(y)), C_a(x)\} = \frac{\delta}{\delta\theta^a(x)}\,\delta_\theta(F_b[\boldsymbol{A}(y)]) \equiv \delta_a F_b. \tag{14.27}$$

For the constraint δ-function we may insert

$$\delta(C_a) = \text{const} \cdot \int \mathcal{D}A_a^0 \exp\Big[\frac{i}{\hbar}\int A_a^0(D\boldsymbol{E})_a\Big]$$

so that

$$Z = \int \mathcal{D}\boldsymbol{E}_a \mathcal{D}A_\mu^a \delta(F_a)\det(\delta_a F_b)\exp\Big[\frac{i}{\hbar}\int(\dot{\boldsymbol{A}}_a\boldsymbol{E}_a - (DA^0)_a E_a - \frac{1}{2}\boldsymbol{E}_a^2 - \frac{1}{2}\boldsymbol{B}_a^2)d^4x\Big], \tag{14.28}$$

where we have partially integrated in the exponent. Next we calculate the Gaussian $\boldsymbol{E}_a$-integral which results in

$$Z = \text{const} \cdot \int \mathcal{D}A_\mu^a \delta(F_a)\det(\delta_a F_b)\exp\Big[\frac{i}{\hbar}\Big((\dot{\boldsymbol{A}}_a - (DA^0)_a)^2 - \boldsymbol{B}_a^2\Big)\Big]$$

Comparing with (14.4) and (14.5) we find the covariant expression for the partition function

$$Z = \text{const} \cdot \int \mathcal{D}A_\mu^a\ \delta(F_a)\det(\delta_a F_b)\ e^{\frac{i}{\hbar}S[A]}. \tag{14.29}$$

In our derivation the gauge conditions F_a depend only on the spatial components of the gauge potential. Recall that $\det(\delta_a F_b)$ is the determinant of the scalar-products of the gradient vectors $\nabla_{\boldsymbol{A}} F_b(\boldsymbol{A})$ with the symmetry-generating vector-fields (generating the θ_a-gauge orbits). We may now assume that F_b also depends on A_0 as long as we guarantee that the determinant keeps this geometric meaning in the enlarged space of the gauge potentials (and not only their spatial components). But also in this enlarged space

$$\frac{\delta}{\delta\theta^a(x)}\delta_\theta F_b = \frac{\delta F_b}{\delta A_\mu^c}\big(\frac{\delta}{\delta\theta^a}\delta_\theta A_\mu^c\big) = (\nabla F_b, X_a), \tag{14.30}$$

where now the gauge transformation may depend on time as well, and hence in $\delta_a F_b$ we must take the gauge variation of all components of A_μ^a. We see that the gauge fixing functions F_a in (14.29) may depend on all components of the gauge potential. Since the action is gauge-invariant, (14.29) still holds and the second equation in (14.27) still defines the object $\delta_a F_b$ appearing in the path integral.

We can derive a more general representation for the transition amplitude than (14.29) by shifting $F_a \to F_a + g_a$, where the functions g_a do not depend on the gauge potential and hence $\delta_a(F_b - g_b) = \delta_a F_b$. Since (14.29) is independent of the gauge choice F_a it is also independent of the functions g_a. Hence (we suppress $\hbar$)

$$\begin{aligned} Z &= \text{const} \cdot \frac{\int \mathcal{D}g G(g) \int \mathcal{D}A\ \delta(F_a - g_a)\det(\delta_a F_b)\ e^{iS[A]}}{\int \mathcal{D}g G(g)} \\ &= \text{const'} \cdot \int \mathcal{D}A\ G(F_a)\det(\delta_a F_b)\ e^{iS[A]}. \end{aligned} \tag{14.31}$$

At this point one can introduce Grassmann-valued fields, so-called Fadeev-Popov ghosts $\eta, \bar\eta$ to represent the determinant of the infinitesimal gauge transformations, so that finally

$$Z[j] = \text{const} \cdot \int \mathcal{D}A\mathcal{D}\eta\mathcal{D}\bar\eta\, G(F_a)\; e^{i\left(S[A]+\int \bar\eta(\delta_a F_b)\eta+\int j^\mu A_\mu\right)}, \tag{14.32}$$

where we have re-introduce the coupling to a conserved current. The constant in front of the path integral is chosen such that $Z[0] = 1$.

Let us see apply this formalism to the Lorentz gauge

$$F_a(A) = \partial_\mu A^\mu_a, \tag{14.33}$$

the infinitesimal gauge variation of which reads

$$\delta_\theta F_b(A) = -\partial^\mu\partial_\mu\theta_b + f_{bcd}\partial_\mu(\theta_c A^\mu_d). \tag{14.34}$$

We strip of the gauge parameter and obtains the following Faddeev-Popov operator,

$$\delta_a F_b = \frac{\delta}{\delta\theta^a(x)}\delta_\theta F_b(A(y)) = \Big(-\delta_{ab}\partial^2 + f_{abc}A^\mu_c(x)\partial_\mu\Big)\delta(x-y).$$

Let us further take

$$G(F_a) = \exp\Big[\frac{i}{2\lambda}\int F_a^2\Big]. \tag{14.35}$$

Finally, writing

$$S[A] = -\frac{1}{4}\int F^a_{\mu\nu}F^{\mu\nu}_a = \frac{1}{2}\int A^\mu_a(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu_a + S_{int}[A], \tag{14.36}$$

where $S_{int}[A]$ contains all the cubic and quartic (self-interacting) terms, the path integral takes the form

$$Z[j] = \text{const} \cdot \int \mathcal{D}A\mathcal{D}\eta\mathcal{D}\bar\eta\; e^{i\left(S_{\text{eff}}[A,\eta,\bar\eta]+\int j^\mu A_\mu\right)}, \tag{14.37}$$

where

$$S_{\text{eff}}[A,\eta,\bar\eta] = S^0_{\text{eff}} + S^{int}_{\text{eff}}. \tag{14.38}$$

We have split S_{eff} into a quadratic term and a term containing higher orders of the fields,

$$\begin{aligned} S^0_{\text{eff}} &= \frac{1}{2}\int A^\mu_a\Big(\eta_{\mu\nu}\partial^2 - (1-\frac{1}{\lambda})\partial_\mu\partial_\nu\Big)A^\nu_a + \int \bar\eta_a(-\partial^2)\eta_a \\ S^{int}_{\text{eff}} &= S_{int}[A] + \int \bar\eta_a(f_{abc}A^\mu_c\partial_\mu)\eta_b. \end{aligned} \tag{14.39}$$

Now we see the effect of the gauge fixing more clearly. Whereas S_0 (the term quadratic in the gauge potential) has zero modes, $S_0[A_\mu = \partial_\mu\lambda] = 0$, and hence cannot be inverted, the effective quadratic term in (14.39) has no zero mode and can be inverted.

14.2 Abelian Gauge Theories

In the Abelian case $f_{abc} = 0$ and the interaction terms are absent. The ghost integral is independent of the gauge potential and chancels in the normalized path integral. Hence

$$Z[j] = \text{const} \cdot \int \mathcal{D}A \, e^{iS^0_{\text{eff}}[A]+i\int j^\mu A_\mu}, \tag{14.40}$$

where

$$S^0_{\text{eff}} = \frac{1}{2}(A^\mu, K_{\mu\nu}A^\nu), \qquad K_{\mu\nu} = \eta_{\mu\nu}\partial^2 - (1-\frac{1}{\lambda})\partial_\mu\partial_\nu. \tag{14.41}$$

Since the operator K has no zero modes we can calculate the Gaussian integral and find

$$Z[j] = \exp\Big[-\frac{i}{2}(j^\mu, K^{-1}_{\mu\nu}j^\nu)\Big] \tag{14.42}$$

for the partition function, where the propagator is easily found to be

$$K^{-1}_{\mu\nu} = \frac{1}{\partial^2}\Big(\eta_{\mu\nu} - (1-\lambda)\frac{1}{\partial^2}\partial_\mu\partial_\nu\Big). \tag{14.43}$$

Common choices for λ are $\lambda = 1$ (Feynman gauge) and $\lambda = 0$ (Landau gauge).

The continuation to the Euclidean sector is achieved by replacing $\boldsymbol{E} \to -i\boldsymbol{E}$, $\boldsymbol{B} \to -\boldsymbol{B}$ and $d^3x \to -id^3x$, so that

$$Z[j] = C \cdot \int \mathcal{D}A \, e^{-S^0_{\text{eff}}[A]+\int jA}, \tag{14.44}$$

where now

$$S^0_{\text{eff}} = \frac{1}{2}(A^\mu, K_{\mu\nu}A^\nu) \quad \text{with} \quad K_{\mu\nu} = -\delta_{\mu\nu}\Delta + (1-\frac{1}{\lambda})\partial_\mu\partial_\nu, \tag{14.45}$$

so that

$$Z[j] = \exp\Big[\frac{1}{2}(j^\mu, K^{-1}_{\mu\nu}j^\nu)\Big]. \tag{14.46}$$

The Euclidean propagator reads

$$K^{-1}_{\mu\nu} = \frac{1}{\Delta}\Big(-\delta_{\mu\nu} + (1-\lambda)\frac{1}{\Delta}\partial_\mu\partial_\nu\Big). \tag{14.47}$$

14.3 The Schwinger model, Part II

After these preparations we are now ready to quantize the bosonic degrees of freedom of the Schwinger model, that is integrate over the 'photon' field. In the following it will be convenient to Hodge-decompose the gauge potential as

$$A_\mu = \epsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\lambda, \tag{14.48}$$

where λ is a pure gauge degree of freedom and drops in gauge invariant expressions. In particular

$$F_{01} = -\Delta\phi \Longrightarrow \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\Delta\phi)^2, \tag{14.49}$$

and the effective action Γ in (12.51) becomes

$$\Gamma[A] = \frac{1}{2}\int \phi\Big(\Delta^2 - \frac{e^2}{\pi}\Delta\Big)\phi. \tag{14.50}$$

The function Φ in (12.39) simplifies to

$$\Phi = \lambda - i\gamma_5\phi. \tag{14.51}$$

Note that both the effective action and the Green function are local in the new fields ϕ and λ.

We shall use the representation (14.29) (or rather its Euclidean continuation) for the path integral, where we choose the Lorentz gauge

$$F = \partial_\mu A^\mu = \Delta\lambda \tag{14.52}$$

and transform variables from A to ϕ, λ. First we note that the Jacobian of the transformation (14.48) is just

$$J = \det\begin{pmatrix} \partial_1 & \partial_0 \\ -\partial_0 & \partial_1 \end{pmatrix} = \det{}^{1/2}\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} = \det(\Delta) \tag{14.53}$$

and second the constraint becomes

$$\delta(F) = \delta(\Delta\lambda) = \frac{1}{\det(\Delta)}\delta(\lambda).$$

The important point is that neither the Jacobian J nor the determinant coming from rewriting the constraint in the new variables depend on the gauge potential and hence they cancel in expectation values against the normalization (here they cancel each other even without normalization). If we compute the expectation value of a gauge invariant operator, say O, which does not depend on the field λ, then the λ-integration is trivial and one obtains

$$\langle O\rangle = \frac{1}{Z[0]}\int \mathcal{D}\phi e^{-\Gamma[\phi]}O[\phi], \quad \text{where} \quad Z[0] = \int \mathcal{D}\phi e^{-\Gamma[\phi]}. \tag{14.54}$$

The most general $2n$-point function (e.g. the two-point function (12.58) are not gauge-invariant but we can built gauge invariant objects out of them, namely operators of the form

$$\exp\Big(i\int_x^y A\Big)\bar\psi(y)M\psi(x), \tag{14.55}$$

or functions of such bilinears. Here M is one of the four matrices Id, γ_5 and γ^μ. The phase factor is needed for the bilinear expression to be gauge invariant (recall that $\psi \to \exp(i\lambda)\psi$ under gauge transformations). Using

$$T\langle 0|\bar{\psi}(y)M\psi(x)|0\rangle = -\langle 0|M_\beta^{\ \alpha}\psi_\alpha(x)\,\bar{\psi}^\beta(y)|0\rangle = -\mathrm{tr}\,MG(x,y) \tag{14.56}$$

one finds

$$\langle e^{ie\int A}\bar{\psi}(y)M\psi(x)\rangle = -\frac{1}{Z[0]}\int\mathcal{D}\phi e^{-\Gamma[\phi]}\,e^{ie\int\epsilon_{\mu\nu}\partial_\nu\phi dx^\mu}\mathrm{tr}\;MG(x,y)|_{\lambda=0}. \tag{14.57}$$

Recalling that ((12.42))

$$G(x,y)|_{\lambda=0} = e^{\gamma_5(e\phi(x)-e\phi(y))}G_0(x-y),\quad \text{where}\quad G_0(\xi) = -\frac{i}{2\pi}\frac{\xi^\mu\gamma_\mu}{\xi^2} \tag{14.58}$$

we see that the spinorial trace in (14.58) vanishes for $M = Id$ and $M = \gamma_5$ and thus

$$\langle J_\pm\rangle = 0,\quad \text{where}\quad J_\pm = \bar{\psi}P_\pm\psi,\qquad P_\pm = \frac{1}{2}(1\pm\gamma_5). \tag{14.59}$$

Similarly, using

$$\begin{aligned}
&T\langle 0|\bar{\psi}(y_1)M\psi(x_1)\cdot\bar{\psi}(y_2)N\psi(x_2)|0\rangle\\
&= M_{\beta_1}^{\ \alpha_1}N_{\beta_2}^{\ \alpha_2}\Big(G_{\alpha_1}^{\ \beta_1}(x_1,y_1)G_{\alpha_2}^{\ \beta_2}(x_2,y_2) - G_{\alpha_1}^{\ \beta_2}(x_1,y_2)G_{\alpha_2}^{\ \beta_1}(x_2,y_1)\Big)\\
&= \mathrm{tr}\,[MG(x_1,y_1)]\,\mathrm{tr}\,[NG(x_2,y_2)] - \mathrm{tr}\,[MG(x_1,y_2)NG(x_2,y_1)]
\end{aligned}$$

one finds for $M = P_-$ and $N = P_+$

$$\begin{aligned}
\langle\bar{\psi}(x)P_-\psi(x)\cdot\bar{\psi}(y)P_+\psi(y)\rangle &= -\frac{1}{Z[0]}\int\mathcal{D}\phi e^{-\Gamma[\phi]}\mathrm{tr}\;P_-G(x,y)P_+G(y,x)\\
&= \frac{1}{Z[0]}\int\mathcal{D}\phi e^{-\Gamma[\phi]}\mathrm{tr}\;P_-\,e^{2\gamma_5[e\phi(x)-e\phi(y)]}G_0^2(x-y) \qquad (14.60)\\
&= -\frac{1}{Z[0]}\frac{1}{4\pi^2(x-y)^2}\int\mathcal{D}\phi\,e^{-\Gamma[\phi]}\;e^{2[e\phi(y)-e\phi(x)]}
\end{aligned}$$

where we have inserted the explicit form (12.42) of G and used that γ_5 anti-commutes with G. Also note that the phase factor is not present in this correlation function. The remaining path integral is Gaussian, that is has the form

$$\frac{1}{Z[0]}\int\mathcal{D}\phi e^{-\Gamma[\phi]+\int j\phi} = e^{\frac{1}{2}(j,Dj)}, \tag{14.61}$$

where the propagator D is determined by the operator appearing in (14.50) and therefore reads

$$D = \frac{1}{\Delta(\Delta-\frac{e^2}{\pi})} = \frac{\pi}{e^2}\Big(\frac{1}{\Delta-e^2/\pi}-\frac{1}{\Delta}\Big). \tag{14.62}$$

D is just the difference of a massive and massless Klein-Gordon propagator. Whereas the Klein-Gordon operator is ultra-violet divergent the effective propagator D is well behaved for $x = y$. Comparing (14.60) and (14.61) we see that $j(z) = 2e\delta(y - z) - 2e\delta(x - z)$ so that

$$\langle J_-(x)J_+(y)\rangle = -\frac{1}{4\pi^2(x-y)^2}\, e^{2e^2\big[D(x,x)+D(y,y)-2D(x,y)\big]}, \tag{14.63}$$

where we have used that D is symmetric in its arguments. For large separations $r = |x - y| \to \infty$ only the massless propagator contributes to $D(x, y)$ and thus (see (12.43))

$$D(x,y) \longrightarrow -\frac{\pi}{e^2}\langle x|\frac{1}{\Delta}|y\rangle = -\frac{1}{2e^2}\log[\mu r]. \tag{14.64}$$

The function $\exp(-4e^2 D(x, y)) \sim \mu^2(x - y)^2$ grows sufficiently fast to cancel the decreasing factor in (14.63) and thus makes the whole expression remain constant for large separations

$$\langle J_-(x)J_+(y)\rangle \longrightarrow -\frac{\mu^2}{4\pi^2}\, e^{4e^2 D(0)}. \tag{14.65}$$

To find the numerical value we must compute $D(0)$. The exact massive propagator is just a Bessel function

$$\langle x|\frac{1}{\Delta - e^2/\pi}|y\rangle = -\frac{1}{2\pi}K_0(er/\sqrt{\pi}) \sim \frac{1}{2\pi}\Big[\log(er/2\sqrt{\pi}) + \gamma\Big] \tag{14.66}$$

where $\gamma = 0.577215$. Together with the massless propagator (12.43) one finds then

$$\langle x|D|y\rangle \sim \frac{\pi}{e^2}\frac{1}{2\pi}\Big[\log(er/2\sqrt{\pi}) + \gamma - \log(\mu r)\Big] = \frac{1}{2e^2}\Big[\log\frac{e}{2\mu\sqrt{\pi}} + \gamma\Big]. \tag{14.67}$$

The only natural mass-scale is the mass of the 'photon', hence we set $\mu = e/\sqrt{\pi}$ and then

$$4e^2\langle x|D|y\rangle \sim -\log(4) + 2\gamma$$

so that finally

$$\langle J_-(x)J_+(y)\rangle \longrightarrow -\frac{e^2}{16\pi^3}\, e^{2\gamma} \tag{14.68}$$

(the overall sign does not agree with the result in the literature?). For completeness we also write down the exact answer

$$\langle J_-(x)J_+(y)\rangle = -\frac{e^2}{16\pi^3}\, e^{2\gamma}\, \exp\Big[2K_0(er/\sqrt{\pi})\Big]. \tag{14.69}$$

Now there is a subtle problem with the result (14.68) or (14.69). For a system with a unique vacuum state the linked cluster property should hold, which states that

$$\langle J_-(x)J_+(y)\rangle \longrightarrow \langle J_-(x)\rangle \cdot \langle J_+(y)\rangle = \langle J_-(0)\rangle \cdot \langle J_+(0)\rangle \tag{14.70}$$

for $|x-y| \to \infty$. In other words the connected 2-point function of J_- and J_+ should decay for large separations. From (14.70) we conclude that

$$\langle J_- \rangle = \frac{e}{4\pi}\frac{1}{\sqrt{\pi}}e^\gamma e^{-i\theta} \quad \text{and} \quad \langle J_+ \rangle = \frac{e}{4\pi}\frac{1}{\sqrt{\pi}}e^\gamma e^{+i\theta}, \tag{14.71}$$

where θ is an arbitrary parameter not fixed by our considerations. Summing the two expectation values yields then

$$\langle \bar{\psi}\psi \rangle = \frac{e}{2\pi}\frac{1}{\sqrt{\pi}}e^\gamma \cos(\theta) \tag{14.72}$$

that is a generically non-vanishing fermionic condensate. On the other hand, in (14.59) we concluded that the expectation values (14.71) and hence the condensate must vanish. What went wrong?

To see what are the problems with the above calculation let use study the zero-energy eigenstates of the Dirac operator. Introducing spherical coordinates

$$x_0 = r\cos(\phi) \quad \text{and} \quad x_1 = r\sin(\phi)$$

the Dirac-operator reads

$$\not{D} = \begin{pmatrix} 0 & e^{-i\phi}(D_r - \frac{i}{r}D_\phi) \\ e^{i\phi}(D_r + \frac{i}{r}D_\phi) & 0, \end{pmatrix}$$

so that the Dirac equation for the zero-energy states $\psi = (\psi_+, \psi_-)$ can be rewritten as

$$A_\phi = -i\partial_\phi \log(\psi_\epsilon) - \epsilon r\partial_r \log(\psi_\epsilon). \tag{14.73}$$

Integrating this equations around a circle or radius R and introducing the electric flux $2\pi\Phi(R) = \oint_R A_\phi d\phi$ through the corresponding disk yields

$$2\pi\Phi(R) = -i\oint \partial_\phi \log(\psi_\epsilon) - \epsilon r\partial_r \oint \log\psi_\epsilon, \tag{14.74}$$

where we have chosen the spherical gauge $A_r = 0$ in the gauge invariant expression (14.72). The first integral on the right hand is just the winding number m of the solutions, e.g. if $\psi \sim \exp(im\phi)$ then it coincides with the angular momentum.
Near the origin a normalizable ψ must be smaller then $1/r$ and since $\Phi(0) = 0$ we find

$$\epsilon = + : \quad (m+1) > 0; \quad \epsilon = - : \quad (m-1) < 0 \Longleftrightarrow \epsilon \cdot m > -1. \tag{14.75}$$

For large radii the wave function must decay more rapidly than $1/r$ and setting $\Phi = \Phi(\infty)$ we obtain

$$\epsilon = + : \quad (\Phi - m) > 1; \quad \epsilon = - : \quad (\Phi - m) < -1 \Longleftrightarrow \epsilon \cdot (\Phi - m) > 1. \tag{14.76}$$

It follows that m and Φ possess the same sign and that $0 \le m < |\Phi| - 1$ and $1 - |\Phi| < m \le 0$ for $\epsilon = +$ and $\epsilon = -$ respectively. Given Φ, the conditions on ϵ and m can be summarized as

$$m\Phi \ge 0, \quad \epsilon \cdot \Phi \ge 0 \quad \text{and} \quad 0 \le |m| < |\Phi| - 1. \tag{14.77}$$

Note that there are only either right- or lefthanded zero-modes, depending on the sign of the total flux, and that the total number of zero modes is just the biggest integer less than $|\Phi|$. For example, for a flux $\Phi = 3.1$ there are 3 zero modes ψ_+, but for $\Phi = 1$ there is no zero mode.

Now, for gauge fields for which the Dirac operator possesses zero modes (12.20) is not equal to (12.22) as we shall see next. Lets assume that the Dirac operator has n zero-modes which we denote by $\psi_j,\ j = 1, \dots, n$. The excited modes we denote by $\psi_k,\ k = n+1, \dots, \infty$). Decomposing the field operators as

$$\psi(x) = \sum_1^n \alpha_j \psi_j(x) + \sum_{n+1}^\infty \beta_k \psi_k(x)$$

and similarly $\bar\psi$ one has

$$\begin{aligned}(\bar\eta, \psi) &= \sum (\bar\eta, \psi_j)\alpha_j + \sum (\bar\eta, \psi_k)\beta_k \\ (\bar\psi, \eta) &= \sum \bar\alpha_j(\psi_j, \eta) + \sum \bar\beta_k(\psi_k, \eta).\end{aligned}$$

Inserting this decomposition into (12.20) and using $\mathcal{D}\psi\mathcal{D}\bar\psi = \mathcal{D}\alpha\mathcal{D}\bar\alpha\mathcal{D}\beta\mathcal{D}\bar\beta$ the integral over the α's can easily be done since the action does not depend on them. One finds

$$\begin{aligned}\int \mathcal{D}\alpha\mathcal{D}\bar\alpha \exp\Big[\sum (\bar\eta, \psi_j)\alpha_j + \bar\alpha_j(\psi_j, \eta)\Big] &= \int \mathcal{D}\alpha\mathcal{D}\bar\alpha\, \frac{1}{n!}\Big[\sum (\bar\eta, \psi_j)\alpha_j + \bar\alpha_j(\psi_j, \eta)\Big]^n \\ = \int \mathcal{D}\alpha\mathcal{D}\bar\alpha \prod \alpha_j\bar\alpha_j \prod (\bar\eta, \psi_j)(\psi_j, \eta) &= \prod_1^n (\bar\eta, \psi_j)(\psi_j, \eta).\end{aligned}$$

The remaining β-integration is performed by shifting

$$\beta_k \longrightarrow \beta_k - \frac{1}{\lambda_k}(\psi_k, \eta) \quad \text{and} \quad \bar\beta_k \longrightarrow \bar\beta_k - \frac{1}{\lambda_k}(\bar\eta, \psi_k),$$

where the λ_k are the (non-zero) eigenvalues of the modes ψ_k (This can be generalized to the situation where the excited modes are scattering states. Then one uses the Greensfunction on the space orthogonal to the zero-modes). After this shift the β integration yields

$$\int \mathcal{D}\beta\mathcal{D}\bar\beta \exp\Big[\sum_1^n \lambda_k\bar\beta_k\beta_k \sum_{n+1}^\infty (\bar\eta, \psi_k)\frac{1}{\lambda_k}(\psi_k, \eta)\Big] = \det{}'(i\not{D})e^{-\int \bar\eta(x)G_e(x,y)\eta(y)},$$

where $\det'$ is the determinant with the zero-eigenvalues omitted and G_e is the Green function of the excited states that is on the space orthogonal to the zero modes

$$i\not{D}G_e(x, y) = \delta(x - y) - \sum \psi_j(x)\psi_j^\dagger(y). \tag{14.78}$$

Inserting all this into the path integral for the partition function we end up with

$$Z[\bar\eta,\eta]=\prod_1^n(\bar\eta,\psi_k)(\psi_j,\eta)\text{det}'(i\not{D})\;e^{-\int\bar\eta G_e\eta} \tag{14.79}$$

and this is the generalization of (12.24) when fermionic zero-modes are present.

Let us now come back to problem of computing the two point functions (14.56) with $M=Id$ and $M=\gamma_5$. We have already seen that the naive calculation, which is valid for gauge fields with no zero-modes, that is for gauge fields with total flux less or equal to 1, gives no contribution. The gauge field with 2 or more zero modes do not contribute either, since Z is higher order in the fermionic current so that after differentiating twice with respect to these currents and setting them afterward to zero on gets a zero-result. So the only contribution comes from the gauge fields with flux between 1 and 2 or -1 and -2. Those have exactly one zero mode ψ_1 and thus

$$\int\mathcal{D}\bar\psi\mathcal{D}\psi\;\bar\psi(x)M\psi(x)=\text{det}'(i\not{D})\text{tr}\,(\bar\psi_1(x)M\psi_1(x)). \tag{14.80}$$

For $M=P_+$ only the right-handed zero mode contributes and thus only gauge potentials with $1<\Phi\leq 2$. For $M=P_-$ only the left-handed zero mode contributes and thus only gauge potentials with $-2\leq\Phi<-1$.
Typical gauge configurations having fermionic zero-modes are the vortex potentials

$$A_\mu=-\frac{\Phi(r)}{r^2}\epsilon_{\mu\nu}x^\nu \tag{14.81}$$

where Φ is a function which vanishes at the origin so that A is regular there and tends to a constant value for large radii $\Phi(r)\longrightarrow\Phi$. The corresponding ϕ in the decomposition (14.48) and field strength read

$$\phi(r)=-\int^r\frac{\Phi(r')}{r'}dr'\sim\Phi\log(r)\quad\text{and}\quad F_{01}=-\Delta\phi=\frac{\Phi'(r)}{r} \tag{14.82}$$

from which follows that the Φ's in (14.82) and (14.74) are the same. For these vortex fields both the primed determinant (after subtracting the determinant of the free Dirac operator) and the classical Maxwell action are finite and so is then the effective action Γ appearing in the bosonic path integral. Thus the functional integration over ϕ's with a given vortex flux should yield a non-zero answer for

$$\langle J_+(x)\rangle=\frac{\int\limits_{1<\Phi\leq 2}\mathcal{D}\phi e^{-\Gamma[\phi]}\text{tr}\,(\bar\psi_1(x)P_+\psi_1(x))}{\int\limits_{-1\leq\Phi\leq 1}\mathcal{D}\phi e^{-\Gamma[\phi]}}, \tag{14.83}$$

where the effective action in the denominator has the form (2.87a) and the one in the numerator contains the classical Maxwell term and the primed determinant. As far as I now, nobody has so far attempted to calculate the remaining path integral over ϕ in the continuum. But we see that our previous naive calculation missed this non-vanishing term.

Similar considerations show that in the correlation function (14.60) the zero-modes drop completely, since for a given gauge potential these modes are either left- or right handed. This is the reason why the naive calculation above yields the correct result for the expectation values (14.68,14.69).

This finishes the technical part of our discussion of the Schwinger model. Most of the results presented have been obtained by Nielsen and Schroer [52]. The Schwinger model on the sphere and the torus have also been studied and the results of these refined calculations agree with (14.71,14.72). So there is no doubt that the Schwinger model shows a breaking of the chiral symmetry (the operator $\bar{\psi}\psi$ transforms non-trivially under global chiral transformations). One may ask what happened to the celebrated Goldstone theorem since on the one hand a continuous $U(1)$ symmetry is broken and on the other hand there is no massless Goldstone boson. The answer to this apparent contradiction comes from the fact that the axial current is not conserved in the Schwinger model, and the derivation of the Goldstone theorem assumes a conserved Noether current. The Schwinger model possesses another quiet interesting property. If we couple the gauge potential to an external current $\mathcal{L} \longrightarrow \mathcal{L} + j^\mu A_\mu$ with $j^0(x) = \rho(x) = q_1\delta(x - x_1) + q_2\delta(x - x_1)$, then the interaction decreases exponentially with the separation $|x_1 - x_2|$ of the two charges, due to the mass of the photon. So the expected long range Coulomb force does not appear. This can only happen if the charges q_1 and q_2 are shielded. The physical mechanism responsible for this charge shielding is the spontaneous pair production. As soon as one tries to separate two 'quarks' (we call the fundamental field ψ quark field to emphasize the analogy to QCD) it is favorable to create a quark pair out of the vacuum and then each of the two created quarks shield one of the originally present quarks. The physical particles of the theory are quark pairs, and not quarks.