Spinning Solitons in Minkowski Space

Dissertation
zur Erlangung des akademischen Grades
doctor rerum naturalium (Dr. rer. nat.)

vorgelegt dem Rat der Physikalisch-Astronomischen Fakultät
der Friedrich-Schiller Universität Jena

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geboren am 23. August 1973 in Rostock

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Tag der letzten Rigorosumsprüfung: 08. Mai 2003

Tag der öffentlichen Verteidigung: 19. Juni 2003

Diese Dissertation entstand zwischen August 1999 und Februar 2003 in der Arbeitsgruppe Quantenfeldtheorie am Theoretisch-Physikalischen Institut der Friedrich-Schiller-Universität Jena.

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Notation And Conventions

In field theory it is common to use units where the speed of light and $\hbar$ are taken to be unity.

The summation convention of Einstein is used, i.e. repeated indices are summed over.

The signature of the metric is $(+, -, -, -)$.

Greek letters μ, ν, ρ, σ refer to space-time and run from 0 to 3, with x^0 being the time coordinate.

Small Latin letters from the middle of the alphabet such as i, j, k denote spatial coordinates and usually run from 1 to 3.

Small Latin letters from the beginning of the alphabet such as a, b, c indicate group indices and run from 1 to $\dim \mathcal{G}$.

A prime denotes differentiation with respect to the argument, whereas a dot over any quantity denotes the time derivative of this quantity.

If not stated otherwise, the following notations are used within this thesis.

A_μ	gauge-potential
A_μ^a	components of non-Abelian gauge-potential
$A = A_\mu \mathrm{d}x^\mu$	gauge connection
$\mathcal{D}_\mu = \partial_\mu - \mathrm{i}A_\mu$	covariant derivative
$\mathcal{D}_\mu = \partial_\mu - \mathrm{i}[A_\mu, \ ]$	covariant derivative for non-Abelian gauge theories, fields in adjoint representation
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$	Abelian field-strength tensor

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc} A^b_\mu A^c_\nu$	components of the non-Abelian field strength tensor
$g_{\mu\nu}$	metric tensor
x, y, z	Cartesian coordinates
ρ, φ, z	cylindrical coordinates
r, θ, φ	spherical coordinates
$\delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$	Kronecker-symbol
$\epsilon_{abc} = \begin{cases} 1 & \text{if } (abc) = (123) \text{ or cycl.} \\ -1 & \text{if } (abc) = (132) \text{ or cycl.} \\ 0 & \text{otherwise} \end{cases}$	complete antisymmetric tensor
ϕ	(complex) scalar field
ϕ^a	components of scalar field in adjoint representation
σ_a	Pauli-matrices $a = 1, 2, 3$
$\sigma_a \sigma_b = \delta_{ab} \mathbb{1} + \mathrm{i} \epsilon_{abc} \sigma_c$	
T_a	generators of gauge group $\mathcal{G}$
$[\mathrm{T}_a, \mathrm{T}_b] = \mathrm{i} f_{abc} T_c \;, \quad \mathrm{Tr}(\mathrm{T}_a \mathrm{T}_b) = K \delta_{ab}$	
$\langle A, B \rangle = \frac{1}{K} \mathrm{Tr}(AB)$	scalar product in the Lie algebra with $A = A^a \mathrm{T}_a$ and $B = B^a \mathrm{T}_a$
$\mathrm{T}_a = \tau_a = \frac{\sigma_a}{2}$	generators for complex doublet representation of $SU(2)$
$(\mathrm{T}_a)_{ik} = -\mathrm{i} \epsilon_{aik}$	generators for real triplet representation of $SU(2)$

Besides Cartesian generators τ_a, ‘spherical’ generators

$$\tau_r = \cos\varphi \sin\theta \, \tau_1 + \sin\varphi \sin\theta \, \tau_2 + \cos\theta \, \tau_3$$

$$\tau_\theta = \cos\varphi \cos\theta \, \tau_1 + \sin\varphi \cos\theta \, \tau_2 - \sin\theta \, \tau_3 \;=\; \partial_\theta \tau_r$$

$$\tau_\varphi = -\sin\varphi \, \tau_1 + \cos\varphi \, \tau_2 \;=\; \frac{1}{\sin\theta} \partial_\varphi \tau_r$$

will be used as well.

1 Introduction

1.1 What Are Solitons?

The first description of a soliton was given by Russell [1] in 1842 and 1843 at the British Association for the Advancement of Science[1]:

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on a horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon ...

This description already gives some of the important features of solitons: they are localised objects, and their shape is preserved during propagation. The possibility of the occurrence of such an object follows from the equations of motion and therefore from the properties of the medium. Nevertheless, everyday experience with water waves is different. If one observes the excitations of water, generated, for instance, by throwing a stone into it, the spatial extension of the excitation is growing. The reason for this behaviour is dispersion, different velocities for different wavelengths. Starting with a localised excitation, which is a superposition of waves with different wavelengths, dispersion leads to the well-known behaviour of growing spatial extension. A phenomenon like the one observed by Russell can only be explained if non-linear effects are taken into account.

[1] this citation follows [2]

The combined effects of dispersion and non-linearities then allow for stationary objects [3]. Concerning theory, it took about 60 years until Korteweg and de Vries found the equation

$$u_t + u_{xxx} + 6u_x u = 0 \tag{1.1}$$

describing the propagation of waves in shallow water in 1895. Here u gives the height of the water above the undisturbed surface. This equation can be derived from the Newtonian equations of motion for continuous media. The Korteweg-de-Vries equation describes the motion in the limit of long wavelengths, i.e. the wavelength is large compared to the depth of the water. The one-soliton solution of (1.1) corresponding to the wave phenomenon observed by Russell is [3]

$$u(x,t) = \frac{\alpha}{2\cosh^2\left[\sqrt{\frac{\alpha}{4}}\,(x - \alpha t - x_0)\right]}\,. \tag{1.2}$$

The propagation of this wave can be seen in figure 1.1, which also shows the constant shape during propagation. Computer experiments have demonstrated that such waves can penetrate each other and emerge undisturbed from this collision. This is illustrated