Functional Renormalisation Group Equations for Supersymmetric Field Theories

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1 Introduction

Quantum field theory [1, 2] is an important part of modern fundamental research. Quantum electrodynamics (QED), developed in the 1940's and the standard model of elementary particles, which was developed in the 1970's, have proven to be very successful. Predictions from QED have been verified experimentally with very high precision and, up to now, the predictions from the standard model have been confirmed by all accelerator experiments.

A fundamental concept of the standard model are symmetries. They led to the classification of the 'elementary particle zoo' in the 1960's. With the help of symmetries the spectrum of 'elementary' hadronic particles could be understood as bound states of just a few basic building blocks, the quarks [3, 4]. Gauge symmetries enforce the existence of gauge bosons, elementary particles that mediate the forces in the standard model.

Although the standard model has been very successful, there are still open questions. To name just a few, these are the hierarchy problem, that the standard model has no dark matter candidate and that it has not been unified with gravity. These are some of the reasons why the standard model is not considered a fundamental theory but rather an effective theory of electroweak and strong interactions. We thus are in need for a theory beyond the standard model. For a review of such theories see e. g. the article by N. Polonsky [5].

With present knowledge supersymmetry, which combines the spacetime symmetry with a symmetry between bosons and fermions, is a promising candidate for an extension of the standard model. Indeed, it is the only known symmetry that allows to combine internal and external symmetries in a nontrivial way. Therefore it is important to gain deeper insight into supersymmetric theories.

Supersymmetry (SuSy) has become a research field in itself and is now an important ingredient in most theories that go beyond the standard model. Supersymmetry predicts that for every elementary particle a superpartner exists. These are particles that have the same quantum numbers as the particles themselves except for the spin. If SuSy is unbroken the superpartners have the same mass as the original particles. Since these superpartners have not been observed yet, supersymmetry has to be broken in nature. If supersymmetry is broken the superpartners can be much heavier than the particles themselves explaining why they have not been found yet in accelerator experiments so far. Up to now there has been no experimental evidence for supersymmetry. However the hope is that it will be found in new experiments done at the LHC at CERN.

For the analysis of supersymmetric extensions of the standard model simpler models are studied, e. g. Wess-Zumino models or supersymmetric sigma models [6, 7, 8]. Wess-Zumino models have a very simple structure since there are no gauge degrees of freedom but only Yukawa interactions. Nevertheless they exhibit all generic properties of supersymmetric theories. Two-dimensional sigma models are very similar to four-dimensional gauge theories which represent an essential part of the standard model. Of special interest are phase transitions, especially the order of the phase transition at critical points.

However, all the models mentioned above are in general not analytically solvable and approximation schemes are needed. Widely used approximation schemes suffer from the problem that they either break supersymmetry explicitly or, if they preserve supersymmetry, the predictions for phase transitions and critical exponents are not correct because fluctuations of light degrees of freedom are not treated properly. For example, the mean field approximation, which is a good approximation for phase transitions in higher dimensions, breaks supersymmetry due to the different treatment of fermions and bosons [9]. The loop calculation can be extended in a supersymmetric way, but it is not possible to obtain results on phase transitions [10, 11].

Non-perturbative results are often obtained using lattice calculations where the spacetime continuum is replaced by a lattice. Although it is a very successful and powerful method, there are still difficulties in formulating supersymmetry on the lattice. One problem is that Lorentz-symmetry is explicitly broken by the lattice implying broken supersymmetry as well. However, in recent years a lot of progress has been made in realising supersymmetry on the lattice, see e. g. [12, 13, 14, 15, 16].

In order to determine the influence of supersymmetry breaking in the lattice calculation on the results, manifestly supersymmetric approximation schemes are needed and should be compared to lattice calculations. Such an approach is provided by the functional renormalisation group equations (FRG) [17, 18]. They deal with the physics of scales and allow to understand the physics at large scales (small momenta) in terms of fundamental interactions at small scales. This is of particular interest in elementary particle physics where it is desired to gain a macroscopic description of atomic nuclei out of the simple laws that govern the fundamental interactions.

The functional renormalisation group equations have been successfully applied to a wide variety of phenomena, ranging from critical phenomena and phase transitions to applications in finite temperature field theory, QCD and quantum gravity, for reviews see [19, 20, 21, 22, 23, 24].

For the description of macroscopic behaviour there exist powerful tools such as statistical descriptions whereas the microscopic physics is often governed by simple laws. In fact, there is a gap between the microscopic and macroscopic description that has to be bridged. The functional renormalisation group allows to integrate out fluctuations in a systematic way. It acts like a microscope where the resolution can be continuously changed.

With the functional renormalisation group correlation functions can be calculated. The latter contain all information about the physical system after the fluctuations have been integrated out. The exact equations are derived as formal identities from the functional integral that defines the theory. The solution of the flow equation corresponds to a trajectory in theory space, that is the space of all action functionals. Different types of these equations have been formulated [25, 26, 27, 28, 29, 30, 31] but the application to non-perturbative systems is hindered by the complexity of the functional differential equations. An exact equation that provides simple access to systematic expansions is a formulation based on the effective action introduced by C. Wetterich [17, 18].

Up to now the extension of the FRG to supersymmetric theories, which is the aim of this thesis, has been pursued only in a very few attempts. In principle, two approaches are possible for such an extension. On the one hand, we could take care of the symmetries with the aid of Ward-Takahashi identities as it has also been done in studies of Yang-Mills theories, see e. g. [18, 22, 32, 33, 34, 35] . On the other hand, we could construct approximations schemes such that supersymmetry is manifestly preserved during the RG flow. We will follow the latter approach.

In this thesis we formulate the flow equations in superspace. This guarantees that supersymmetry will not be broken by the regulator or the truncation. This ansatz has not been pursued in great detail in earlier studies, in most cases only perturbative results have been obtained. First steps in extending the FRG to supersymmetry have been accomplished by F. Vian and M. Bonini [36, 37, 38], B. Geyer and S. Falkenberg [39] as well as by K. Aoki and co-workers [40]. Applications of non-perturbative renormalisation group methods on supersymmetric theories can be found in the papers by S. Arnone and co-workers [41, 42]. More recently, a general theory of scalar superfields which include the Wess-Zumino model with a Polchinsky-type RG has been formulated by O. Rosten [43, 44]. A Wilson effective action for Wess-Zumino models by perturbative iteration of the functional RG has been formulated by H. Sonoda and K. Ulker [45].

This work is organised as follows: In chapter 2 the basic facts of quantum field theory are collected and the functional renormalisation group equations are derived. Chapter 3 gives a short introduction to the main concepts of supersymmetry that are used in the subsequent chapters. In chapter 4 the functional RG is employed for a study of supersymmetric quantum mechanics, a supersymmetric model which was studied intensively in the literature. A lot of results have previously been obtained with different methods and we compare these to the ones from the FRG. We investigate the $\mathcal{N} = 1$ Wess-Zumino model in two dimensions in chapter 5. This model shows spontaneous supersymmetry breaking and an interesting fixed-point structure. Chapter 6 deals with the three dimensional $\mathcal{N} = 1$ Wess-Zumino model. Here we discuss the zero temperature case as well as the behaviour at finite temperature. Moreover, this model shows spontaneous supersymmetry breaking, too. In chapter 7 the two-dimensional $\mathcal{N} = (2, 2)$ Wess-Zumino model is investigated. For the superpotential a non-renormalisation theorem holds and thus guarantees that the model is finite. This allows for a direct comparison with results from lattice simulations.

The compilation of this work is solely due to the author. However, parts of the results have been obtained in collaboration with colleagues from research groups in Jena, Münster and Southampton. The octave program used to calculate the exact values of the energy of the first excited state in chapter 4 has been provided by A. Wipf. The perturbative calculations of the propagator in chapter 7 have been done by G. Bergner (now University of Münster). The Python program used to calculate the momentum-dependent wave-function renormalisation has been developed by T. Fischbacher (University of Southampton).

2 Functional renormalisation group

In this chapter we sketch the main aspects of the functional renormalisation group (FRG). As the FRG is formulated in Euclidean space-time, we will only discuss this case. For the case of a Minkowskian space-time the reader is referred to the numerous textbooks on QFT, for example the one by M. E. Peskin and D. V. Schroeder [1], S. Weinberg [2, 46] or J. Zinn-Justin [47]. For an introduction to critical phenomena and renormalisation group, see the textbook by J. Cardy [48], for reviews on the FRG see e. g. the paper by J. Berges, N. Tetradis and C. Wetterich [17] or H. Gies [18].

2.1 Basics of QFT

The conventions in this chapter follow [1, 18] if not stated otherwise. The basic objects in quantum field theory are correlation functions as they contain all physical information about the theory. The correlators or *n-point functions* are defined as the product of n fields located at different points in space-time averaged over the quantum fluctuations, i. e. all possible field configurations. In Euclidean field theories, the weight of a field configuration is the exponentiated action

$$\langle \varphi(x_1)\dots\varphi(x_n)\rangle = \mathcal{N}\int \mathcal{D}\varphi\ \mathrm{e}^{-S[\varphi]}\,\varphi(x_1)\dots\varphi(x_n) \tag{2.1}$$

with normalisation constant $\mathcal{N}$. In the following discussion we will concentrate on scalar fields, afterwards we will discuss the generalisation for fermionic fields.

All n-point correlation functions can be obtained from the generating functional $Z[J]$ with J being an external source. The generating functional is defined through

$$Z[J] = \int \mathcal{D}\varphi\ \mathrm{e}^{-S[\varphi]+\int_x J\varphi} \tag{2.2}$$

with the shorthand notation $\int_x J\varphi = \int \mathrm{d}^d x\, J(x)\varphi(x)$ for the external source term. Functional differentiation with respect to the external source yields

$$\langle \varphi(x_1)\dots\varphi(x_n)\rangle = \frac{1}{Z[0]}\left(\frac{\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)}\right)\Bigg|_{J=0}. \tag{2.3}$$

With the generating functional $Z[J]$ another important quantity, the generating functional $W[J]$ of the *connected* n-point functions, is defined as $W[J] = \ln(Z[J])$. The Legendre transformation of $W[J]$ yields the *effective action*. It is defined as

$$\Gamma[\phi] = \sup_J \left(\int_x J\phi - W[J] \right). \tag{2.4}$$

Because of the properties of the Legendre transformation, the effective action is always a convex functional. For a detailed discussion of the effective action and its properties see [49]. Its maximum at $J = J_{\text{sup}}$ gives the vacuum expectation value of the microscopic or classical field φ:

$$0 = \frac{\delta}{\delta J(x)} \left(\int J\phi - W[J] \right) \quad \Rightarrow \quad \phi = \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J} = \langle \varphi \rangle_J. \tag{2.5}$$

The macroscopic field ϕ is the expectation value of the microscopic field φ in the presence of the source J. The equations of motion for the macroscopic field read

$$\frac{\delta\Gamma[\phi]}{\delta\phi} = -\int_y \frac{\delta W[J]}{\delta J(y)} \frac{\delta J(y)}{\delta\phi(x)} + \int_y \frac{\delta J(y)}{\delta\phi(x)} \phi(y) + J(x) = J(x) \tag{2.6}$$

For constant fields the effective action is an extensive quantity and after scaling out the volume, the *effective potential* is given by [50]

$$V_{\text{eff}}(\phi) = V_d^{-1} \cdot \Gamma[\phi]. \tag{2.7}$$

As the effective potential is the part of the effective action that contains no derivatives, it is a convex function as well. For a vanishing source, the effective action coincides with the vacuum energy. $V_{\text{eff}}(\phi)$ therefore is the energy density of the corresponding state. If symmetries are not spontaneously broken (cf. section 2.3) the vacuum state of the theory is given by the absolute minimum of the effective potential. We will discuss the case of spontaneously broken symmetry below.

The effective action is the generating functional of the one-particle-irreducible correlation functions. This means that the effective action contains the complete information about the quantum field theory. For example the vacuum state is given by the minimum of the effective potential, whether symmetries of the Lagrangian are preserved or not depends on the location of the minima. The second derivative of the effective action gives the inverse propagator and the poles of the propagator determine the masses of the particles. From higher-order derivatives of the effective action the one-particle-irreducible amplitudes can be calculated which yield the S-matrix elements.

From the generating functional an equation for the effective action can be obtained:

$$\mathrm{e}^{-\Gamma[\phi]} = \int \mathcal{D}\varphi \; \mathrm{e}^{-S[\phi+\varphi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi} \varphi} \tag{2.8}$$

This equation can only be solved exactly for very special cases, e. g. the Schwinger model.

A very successful approximation is a vertex expansion which leads to the Dyson-Schwinger equations [51, 52, 53], which consist of infinitely many coupled integral equations. They are the equations of motion for the Green functions. For reviews on the Dyson-Schwinger equations see e. g. the works by R. Alkhofer and L. v. Smekal [54] or C. Fischer [55].

In this work we follow a different approach based on the concept of renormalisation.

2.2 The Renormalisation Group

The name *renormalisation group* (RG) has been invented in the 1950s [56, 57], as there was hope that all fundamental physics could be expressed through symmetry and group theory rather than dynamics. At first it was applied to the high energy behaviour of renormalised quantum electrodynamics. K. Wilson realised that it could be put to work for a much larger field of applications, namely the field of critical phenomena [25, 26, 58]. Today it is used for a large class of physical problems such as critical phenomena with long-distance correlations or fluid turbulence.

Wilson's idea was to start at a microscopic theory at large momentum scale Λ and to integrate out the fluctuations momentum shell by momentum shell. This leads to scale-dependent actions that are connected through continuous RG transformations. The RG flow describes how the scale-dependent couplings change under the RG transformation, see e. g. the review by K. G. Wilson and J. B. Kogut [27] for a discussion of the renormalisation group and critical phenomena. For a historical inspired introduction to the RG the reader is referred to Wilson's Nobel Prize lecture [59] and the review by M. Fisher [60].

The renormalisation idea involves a reexpressing of parameters $\mathcal{K}$ of the theory through new parameters $\mathcal{K}'$ without changing its physical content. This was first introduced by Kadanoff [61, 62]. Such a transformation has the form $\{\mathcal{K}'\} = \mathcal{R}\left(\{\mathcal{K}\}\right)$ with $\mathcal{R}$ depending on the transformation and the rescaling parameter. At a fixed point of the transformation $\{\mathcal{K}\} = \{\mathcal{K}^*\}$ and for $\mathcal{R}$ differentiable at the fixed point the transformations can be linearised around the fixed point,

$$\mathcal{K}'_a - \mathcal{K}^*_a = \sum_b T_{ab}(\mathcal{K}_b - \mathcal{K}^*_b), \tag{2.9}$$

with $T_{ab} = \left(\partial\mathcal{K}'_a/\partial K_b\right)\Big|_{\mathcal{K}=\mathcal{K}^*}$. The eigenvalues of T are denoted by λ^I. The left eigenvectors are denoted by $\{e^I\}$ such that

$$\sum_a e^I_a T_{ab} = \lambda^I e^I_b. \tag{2.10}$$

In general, the matrix T does not need to be symmetric and left and right eigenvectors do not need to be identical.

The *scaling variables* $u_I \equiv \sum_a e^I_a(\mathcal{K}_a - \mathcal{K}^*_a)$ play an important role in the description of critical phenomena. They transform multiplicatively at the fixed point,

$$u'_I = \lambda^I u_I. \tag{2.11}$$

To distinguish between different kinds of fixed points, the notation $\lambda_I = b^{\theta_I}$ is introduced, where b is the rescaling parameter and θ_I is the *renormalisation group eigenvalue*. If $\theta_I > 0$, u_I is called a *relevant* direction, because repeated renormalisation iterations will drive the system away from the fixed point. For $\theta_I < 0$ u_i is called *irrelevant*, as it will tend to zero during the RG transformations. For $\theta_I = 0$ u_I is called *marginal* and it cannot be decided from the linear approximation whether u_I is driven towards the fixed point or away from it.

For an N-dimensional system near the fixed point which has n relevant eigenvalues there are $N - n$ irrelevant directions. They form an $(N - n)$-dimensional hypersurface. All points on this hypersurface are attracted towards the fixed point. This surface is called *critical hypersurface* and the long-distance behaviour of systems whose parameters sit on this surface is controlled by the fixed point. For the system to end up in this hypersurface, a fine tuning of the n relevant directions is required.

All critical models that flow into the same fixed point make up a *universality class*, i. e. they show the same quantitative behaviour near a phase transition. This behaviour is governed by the long-range fluctuations and is independent of the details of the specific system. Universality means that near a fixed point the behaviour does not depend on these details.

The rescaling factor b depends on the RG transformation used and not on the model. A description of the critical behaviour, in which b does not enter explicitly, is given in terms of the *β-functions*. For this, an infinitesimal transformation with $b = 1 + \delta l$ and $\delta l \ll 1$ is considered. This leads to an infinitesimal transformation of the couplings,

$$\mathcal{K}_a \to \mathcal{K}_a + \frac{d\mathcal{K}_a}{dl}\delta l + O(\delta l^2), \tag{2.12}$$

and the RG transformations can be written in infinitesimal form as

$$\frac{d\mathcal{K}_a}{dl} = -\beta_a\left(\{\mathcal{K}\}\right) \tag{2.13}$$

with β_a the *renormalisation group β-function*. The fixed points are the zeros of the β-function and the matrix T takes the form

$$T_{ab} = \delta_{ab} - \frac{\partial \beta_a}{\partial \mathcal{K}_b}\delta l. \tag{2.14}$$

The renormalisation group eigenvalues are

$$(1 + \delta l)^{\theta_I} \simeq 1 + \theta_I \delta l. \tag{2.15}$$

Therefore they are given by the eigenvalues of the matrix $-\partial\beta/\partial\mathcal{K}_b$ at the zero of the β-function.

One form that incorporates the idea of the RG equations is the Callan-Symanzik equation [63, 64], which is a differential equation for the evolution of the n-point correlation functions under variation of an energy scale parameter.

Other RG equations have been derived by F. J. Wegner and A. Houghton [28] as well as by J. Polchinski [30]. Here the approach based on the effective average action as introduced by C. Wetterich [65] is followed.

Two fixed points of Ising-like systems[1] are of special interest in the following chapters. One of these is the so-called Gaußian fixed point, which describes a free, non-interacting theory. Its name is derived from the fact that this fixed point has a Gaußian probability distribution. For a detailed discussion of this type of fixed point see the textbook by J. Zinn-Justin [47].

In space-times with less than four dimensions also a nontrivial fixed point exists, called the Wilson-Fischer fixed point [66]. It has been found by an *ε-expansion*, in which the dimension of space-time is taken to be a continuous parameter. The vicinity of four dimensions is explored by taking the deviation from four dimensions, $\varepsilon = 4 - d$, as an expansion parameter.

2.3 Spontaneous symmetry breaking

Systems that exhibit spontaneous symmetry breaking are systems whose dynamics are invariant under some symmetry but the ground state is not. The most prominent example is a ferromagnet at low temperature. Its Hamiltonian is rotationally invariant. In an external magnetic field the elementary magnets are oriented along the magnetic field lines and keep their orientation even after the external field is turned off such that the spherical symmetry of the material is broken. If the material is heated above a specific temperature the orientation of the elementary magnets is lost and the spherical symmetry is restored.

For a system with spontaneous symmetry breaking, the potential in the Lagrangian does not have one uniquely determined minimum, instead it has degenerate minima with the same energy. The effective potential in this case is not strictly convex anymore but is flat between degenerate minima.

For every continuous global symmetry in $d > 2$ that is spontaneously broken *Goldstone's theorem* states that there must be a massless particle contained in the theory [67]. If the massless particles are bosons, they are called *Goldstone bosons*. For example, pions can be interpreted as (approximate) Goldstone bosons, see e. g. [46] for a discussion.

A proof of this theorem can be found in [1, 68]. In general, a broken global symmetry leads to a Goldstone mode with the same quantum numbers as the generator of the symmetry. As we shall see in the next chapter, this implies that the Goldstone mode for supersymmetry breaking is fermionic [69].

[1] The Ising model is a simple model that is used to describe a ferromagnet.

2.4 Derivation of the flow equation

To compute the effective action in the FRG approach the quantum fluctuations are integrated out in successive momentum shells [65]. The effective average action generalises the block spin picture introduced by Kadanoff [62] to continuous space [70]. The aim of this section is to construct an equation for an interpolating action, the effective average action Γ_k, with a momentum-shell parameter k. We restrict ourselves to bosonic degrees of freedom, the generalisation to fermionic or gauge degrees of freedom is straightforward as we shall discuss below.

The interpolating action has to fulfil the conditions $\Gamma_{k\to\Lambda} = S_{\text{bare}}$ and $\Gamma_{k\to 0} = \Gamma$ and it is constructed from a scale dependent generating functional

$$Z_k[J] \equiv \int \mathcal{D}\varphi \; \mathrm{e}^{-S[\varphi]+\int_x J\varphi - \Delta S_k[\varphi]} \, . \tag{2.16}$$

The scale-dependent cutoff action is chosen to be

$$\Delta S_k[\varphi] = \frac{1}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\varphi(-q)R_k(q)\varphi(q). \tag{2.17}$$

R_k is a regulator function and is required to have the following properties:

- $R_k(q) \to 0$ for $k \to 0$ so that $W_{k\to 0}[J] = W[J]$
- $R_k(q) \to \infty$ for $k \to \Lambda$ so that $\Gamma_{k\to\Lambda} = S[\phi]$
- $R_k(q) > 0$ for $q^2 \to 0$ so that R_k serves as an infrared regulator

Typical bosonic regulators are

$$R_k = \frac{q^2}{\mathrm{e}^{q^2/k^2} - 1} \quad \text{or} \quad R_k = \left(k^2 - p^2\right)\theta\left(\frac{k^2}{p^2} - 1\right). \tag{2.18}$$

With this scale-dependent cutoff action we introduce a modified Legendre transformation in order to obtain the scale-dependent effective action

$$\Gamma_k[\phi] = -W_k[J] + \int_x J\phi - \Delta S_k[\phi] \tag{2.19}$$

with the macroscopic field ϕ defined by

$$\phi(x) = \frac{\delta W_k[J]}{\delta J(x)} = \langle \varphi(x) \rangle_J . \tag{2.20}$$

Taking the derivative $\partial_t = k\partial_k$ with respect to the 'RG-time' $t = \ln\left({}^{k}/_{\Lambda}\right)$, yields

$$\partial_t \Gamma_k[\phi] = -\partial_t W_k[J] + \partial_t \int_x J\phi - \partial_t \Delta S_k[\phi] \tag{2.21}$$

The source J is scale-independent and we calculate the derivative of $W_k[J]$ in the following:

$$\partial_t W_k[J] = \partial_t \ln Z_k[J] = \frac{1}{Z_k[J]} \int \mathcal{D}\varphi \, \frac{1}{2} \int_q \varphi \, \partial_t R_k \varphi \, \mathrm{e}^{-S[\varphi]+\int_x \varphi J - \Delta S_k[\varphi]} \tag{2.22}$$
$$= \frac{1}{2Z_k[J]} \int_q \partial_t R_k \int \mathcal{D}\varphi \; \varphi\varphi \, \mathrm{e}^{-S[\varphi]+\int_x \varphi J - \Delta S_k[\varphi]} = \frac{1}{2Z[J]} \int_q \partial_t R_k \frac{\delta^2 Z[J]}{\delta J \delta J} = \mathrm{e}^{-W_k} \frac{1}{2} \int_q \partial_t R_k \frac{\delta^2 \, \mathrm{e}^{W_k}}{\delta J \delta J}$$

Where the last term can be computed

$$\frac{\delta^2 \, \mathrm{e}^{W_k}}{\delta J \delta J} = \frac{\delta}{\delta J} \left(\mathrm{e}^{W_k} \frac{\delta W_k}{\delta J} \right) = \mathrm{e}^{W_k} \frac{\delta W_k}{\delta J} \frac{\delta W_k}{\delta J} + \mathrm{e}^{W_k} \frac{\delta^2 W_k}{\delta J \delta J} \tag{2.23}$$

Inserting this back into the equation above yields

$$\partial_t W_k[J] = \mathrm{e}^{-W_k} \frac{1}{2} \int_q \partial_t R_k \left(\mathrm{e}^{W_k} \frac{\delta W_k}{\delta J} \frac{\delta W_k}{\delta J} + \mathrm{e}^{W_k} \frac{\delta^2 W_k}{\delta J \delta J} \right) = \partial_t \Delta S_k + \frac{1}{2} \int_q \partial_t R_k \frac{\delta^2 W_k}{\delta J \delta J}. \tag{2.24}$$

With this the effective action reads

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int_q \partial_t R_k \frac{\delta^2 W_k}{\delta J \delta J}. \tag{2.25}$$

Now the term $\frac{\delta^2 W_k}{\delta J \delta J}$ is expressed through the effective action. For this we need the functional derivative of the effective action which yields the equation of motion

$$\frac{\delta \Gamma_k[\phi]}{\delta \phi(x)} = -\int_y \frac{\delta W_k[J]}{\delta J(y)} \frac{\delta J(y)}{\delta \phi(x)} + \int_y \frac{\delta J(y)}{\delta \phi(x)} \phi(y) + J(x) - \left(R_k \phi\right)(x) = J(x) - \left(R_k \phi\right)(x). \tag{2.26}$$

Solving the above equation for $J(x)$ and taking a functional derivative yields

$$\frac{\delta J(x)}{\delta \phi(y)} = \frac{\delta \Gamma[\phi]}{\delta \phi(x) \delta \phi(y)} + R_k(x, y). \tag{2.27}$$

Together with the identity

$$\delta(q - q') = \frac{\delta \phi(q)}{\delta \phi(q')} = \frac{\delta}{\delta \phi(q')} \frac{\delta W_k[J]}{\delta J(q)} = \int_{q''} \frac{\delta^2 W_k[J]}{\delta J(q) \delta J(q'')} \frac{\delta J(q'')}{\delta \phi(q')} \tag{2.28}$$

this leads to

$$\delta(q - q') = \int_{q''} \frac{\delta^2 W_k[J]}{\delta J(q) \delta J(q'')} \left(\frac{\delta^2 \Gamma_k[\phi]}{\delta \phi(q'') \delta \phi(q)} + R_k \right) \tag{2.29}$$

such that

$$\frac{\delta^2 W_k[J]}{\delta J \delta J} = \left(\frac{\delta^2 \Gamma_k[\phi]}{\delta\phi\delta\phi} + R_k\right)^{-1} . \tag{2.30}$$

With this the flow equation reads

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\int_q \partial_t R_k \left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1} = \frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k \left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1}\right]. \tag{2.31}$$

In the denominator the regulator R_k serves as an infrared regulator as it suppress the massless modes whereas the term $\partial_t R_k$ in the numerator serves as an ultraviolet regulator.

As stated above, the generalisation of equation (2.31) to a number of scalar fields and to fermionic fields is straightforward [71, 72, 73, 74]. In this case the trace is taken not only in momentum space but over all internal and external indices as well. For fermionic theories it has to be taken into account that the first derivative acts from the left and the second from the right. Also, instead of the trace the supertrace has to be taken. Thus the flow equation reads

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_t R_k\right] \quad \text{with} \quad \left(\Gamma_k^{(2)}\right)_{ab} = \frac{\overrightarrow{\delta}}{\delta\phi_a}\Gamma_k \frac{\overleftarrow{\delta}}{\delta\phi_b} \tag{2.32}$$

where the indices a, b summarise field components, internal and Lorentz indices, as well as space-time or momentum coordinates.

In a block-matrix notation for the bosonic and fermionic sector the scale-dependent propagator can be written as

$$\left(\Gamma_k^{(2)} + R_k\right)^{-1} = \begin{pmatrix} G_{BB} & G_{BF} \\ G_{FB} & G_{FF} \end{pmatrix}. \tag{2.33}$$

The regulator does not mix bosonic and fermionic degrees of freedom. In block-matrix notation it reads

$$R_k = \begin{pmatrix} R_k^{BB} & 0 \\ 0 & R_k^{FF} \end{pmatrix}. \tag{2.34}$$

Thus, we obtain the following result for the flow equation:

$$\partial_k \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\, G_{BB}\partial_k R_k^{BB} - \frac{1}{2}\,\mathrm{Tr}\, G_{FF}\partial_k R_k^{FF} \tag{2.35}$$

2.5 Properties of the flow equation

The flow equation has a simple one-loop structure, but in contrast to perturbation theory the fully dressed propagators enter and not only the bare ones. The one-loop structure is a consequence of the cutoff action being quadratic in the fields [75].

The flow equation is derived from the generating functional which is usually taken as the starting point to define a quantum field theory (QFT). As already stated in section 2.1 the effective action contains all the information about the quantum field theory. Therefore it is also possible to use the flow equation and initial conditions as the starting point for the field theory because the flow equation defines a trajectory to the full quantum effective action.

This trajectory lies in the so-called *theory space*, that is the space of all action functionals spanned by all possible invariant operators of the fields. The trajectory is determined by the choice of the regulator, which is a manifestation of the RG scheme dependence. Note that the trajectory is a non-universal quantity. As long as the cutoff Λ can be removed, however, the endpoint is unique and independent of the regulator if no approximations are made. Approximations introduce a regulator dependence of the infrared observables, but for a good approximation this dependence is small.

2.6 Truncations

It is in general not possible to solve the flow equation analytically, and therefore approximations have to be employed. The most common ones [17, 18] are listed in this section.

The *vertex expansion* is an expansion in the number of fields which reads

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \dots \mathrm{d}^d x_n \Gamma_k^{(n)}(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n). \tag{2.36}$$

This approximation yields flow equations for the vertex functions $\Gamma_k^{(n)}$ that interpolate between the bare and the fully dressed vertices.

Another possibility is the *operator expansion*. The effective action is made up from operators with increasing mass dimensions. A particular type of this kind of expansion is the *derivative expansion* [76, 77], which is an expansion in powers of the momentum. For scalar field theories it reads

$$\Gamma_k[\phi] = \int \mathrm{d}^d x \left(V_k(\phi) + \frac{1}{2} Z_k(\phi) \left(\partial_\mu \phi\right)^2 + O\left(\partial^4\right) \right). \tag{2.37}$$

V_k denotes the so-called effective potential, Z_k is the wave-function renormalisation.

In order to obtain a good approximation already from the first terms of the expansion, higher derivative operators must have only a small influence compared to the operators of order one. For this to be true the anomalous dimension η of the quantum field, that is the deviation of the scaling law from the one expected from dimensional analysis, has to be small.

The lowest order in the derivative expansion is called the *local potential approximation* (LPA). To this order only the potential is taken to be scale-dependent. In addition, at the next-to-leading order (NLO) a wave function renormalisation is taken into account. From the wave-function renormalisation the anomalous dimension can be calculated with the relation

$$\eta = -\partial_t \ln Z_k. \tag{2.38}$$

In this work, mostly a derivative expansion is employed. However, we shall perform an expansion in terms of super-covariant derivatives in order to preserve supersymmetry (see section 4.2.2).

2.7 Spectrally adjusted flows

Spectrally adjusted flows are used mostly in gauge theories [22, 78, 79]. As we will need it later on the main ideas are shortly sketched. For a more detailed discussion see e. g. [78]. Any truncation selects a hypersurface in the space of all actions. A good truncation is one whose trajectory in the hypersurface is close to the exact RG trajectory projected onto the hypersurface.

The regulator can be improved with respect to this property if the full second functional derivative of the effective action – evaluated in the presence a background field – instead of the squared momentum is used in the argument. This leads to an improvement because the spectrum is not fixed but adjusted during the flow. By including the full $\Gamma_k^{(2)}$ the regulator is adjusted to the flow of the spectrum.

2.8 Recovering perturbation theory

The FRG contains all orders of perturbation theory [75, 80, 81]. To see this, the effective action is expanded in a perturbation series

$$\Gamma_k = S + \sum_n \Delta\Gamma_k^{n\text{–loop}}. \tag{2.39}$$

Considering just the classical action on the right hand side, the flow equation reads

$$\partial_t \Gamma_k^{\text{one–loop}} = \frac{1}{2} \operatorname{Tr} \frac{1}{S^{(2)}[\phi] + R_k} \partial_t R_k = \frac{1}{2} \operatorname{Tr} \partial_t \ln\left(S^{(2)}[\phi] + R_k\right) \tag{2.40}$$

Integration with respect to k yields

$$\Delta\Gamma_k^{\text{one-loop}}[\phi] = \Delta\Gamma_\Lambda^{\text{one-loop}}[\phi] + \int_\Lambda^k \frac{dk'}{k'} \partial_t' \Gamma_{k'}^{\text{one-loop}}[\phi] \tag{2.41}$$

Inserting the above expression leads to

$$\Delta\Gamma_k^{\text{one-loop}}[\phi] = \Delta\Gamma_\Lambda^{\text{one-loop}}[\phi] + \frac{1}{2}\operatorname{Tr}\left[\ln\left(S^{(2)}[\phi] + R_k\right)\right] - \operatorname{Tr}\left[\ln\left(S^{(2)}[\phi] + R_\Lambda\right)\right] \tag{2.42}$$

The cutoff-dependent terms regularise the expression whereas the k-dependent term is finite. Renormalisation implies that the scale-dependent part $\Delta\Gamma_k$ is independent of the cutoff Λ. In this scheme it corresponds to adjusting the Λ independence of $\Delta\Gamma_k$ with $k \neq \Lambda$ (*regularisation*) and fixing the Λ independent parts of $\Delta\Gamma_\Lambda$ (*renormalisation conditions*).

Higher loop orders can be calculated in a similar fashion [75, 80].

3 Basics of supersymmetry

In the early 1970s supersymmetry (SuSy) was invented by Golfand and Likhtman [82], followed by Akulov and Volkov [83, 84] and independently in the context of string theory as a symmetry of two-dimensional world-sheet theory [85, 86, 87, 88]. Later it was realised that SuSy could be a symmetry of four-dimensional quantum field theory and might be important for particle physics.

One reason for introducing supersymmetry is that divergences due to radiative corrections are less severe in supersymmetric theories because of cancellations between bosonic and fermionic loops. Supersymmetry provides a dark matter candidate and might solve the hierarchy problem. Nevertheless SuSy cannot be the full answer since it has to be broken at low energy scales.

Supersymmetry became popular when it was realised by Haag, Łopuszański and Sohnius [89] that it allows to circumvent the prerequisites of the *Coleman-Mandula theorem* [90]. This theorem states that in a theory with a non-trivial scattering matrix in more than $1 + 1$ dimensions the only possible conserved quantities that transform as tensors under the Lorentz group are the generators of the Poincaré group and generators of internal symmetries. Haag, Łopuszański and Sohnius proved [89] that fermionic symmetry operators allow for a unification of space-time and internal symmetries.

The main ideas of supersymmetry will be sketched here without going too deep into the technical details, only general aspects of supersymmetry are presented. There are a lot of excellent textbooks, review articles and lecture notes available, for example the review by M. F. Sohnius [91], the textbooks by S. Weinberg [92] and P. West [93], the lecture notes by A. Wipf [94, 95] and A. Bilal [96] or an article by Y. Shadmi [97]. All these articles were used for this chapter, and the reader is referred to them for a more thorough introduction to supersymmetry. The technical details of the specific models that are investigated are assembled in the respective chapters.

3.1 Supersymmetry algebra

The supersymmetry algebra enlarges the Poincaré algebra by generators Q_i and $\bar{Q}_i$, called supercharges, with $i = 1 \dots \mathcal{N}$. The SuSy generators transform as spinors under the Lorentz group, obey anticommutation relations among each other and commute with translations. The SuSy generators transform bosons into fermions and vice versa:

$$Q\,|\text{boson}\rangle = |\text{fermion}\rangle \quad \text{and} \quad Q\,|\text{fermion}\rangle = |\text{boson}\rangle \tag{3.1}$$

One of the generators Q and $\bar{Q}$ lowers the spin by 1/2 the other raises it by 1/2. The anticommutator of two successive SuSy transformations Q_i and $\bar{Q}_i$ acts as a translation.

The supersymmetry algebra contains the Poincaré algebra and as each irreducible representation of the Poincaré algebra corresponds to a particle, each irreducible representation of a supersymmetry algebra corresponds to several particles that are related by supersymmetry transformations. All these particles form a supermultiplet. Successively applying all generators Q_1 to $Q_{\mathcal{N}}$ on the particle with the largest spin the supermultiplet can be constructed. Without gravity, the largest spin in a renormalisable quantum field theory is one, restricting the number of supercharges in four dimensions to $\mathcal{N} \leq 4$. For a theory with gravity we have $\mathcal{N} \leq 8$ because gravity cannot consistently couple to spins larger than two.

Because the supercharges commute with the generator of translations, $[Q, P^2] = 0$, all particles in a supermultiplet have the same mass. The energy of the particles is always non-negative and if supersymmetry is unbroken the ground state energy is always zero. A supermultiplet contains the same number of bosonic and fermionic degrees of freedom. For proofs see e. g. [96].

In the following mostly the $\mathcal{N} = 1$ scalar multiplet in various dimensions is considered. It contains a bosonic field ϕ which can be real or complex depending on the space-time dimension, a Majorana fermion ψ and an auxiliary field F. The latter is called an auxiliary field because it has algebraic equations of motion. An action that contains the auxiliary field is called an *off-shell* action because the supersymmetry algebra closes without taking into account the equations of motion. An action where the auxiliary field is integrated out is called an *on-shell* action because the supersymmetry algebra closes only when the equations of motion are used.

3.2 Superspace

Superspace, which was introduced by A. Salam and J. Strathdee [98], is a formalism in which supersymmetry is inherently manifest. Analogue to three-dimensional Euclidean space which is extended to four-dimensional Minkowski space for Lorentz invariant theories, Minkowski space (or Euclidean space) is extended to superspace for supersymmetric theories. In this section only $\mathcal{N} = 1$ superspace which is generated by one supercharge is discussed. For the formulation of a superspace with two supercharges see e. g. [93] and appendix E.1.

The elements of superspace are *superfields* which combine the components of the supermultiplet. These are fields $\Phi(x, \theta, \bar{\theta})$ that depend on the space-time coordinates x and Grassmann variables θ and $\bar{\theta}$. Therefore in superspace anticommuting coordinates are added to the commuting coordinates of space-time. In general the SuSy algebra is reducible. In order to reduce the degrees of freedom in a superfield, various constraints are applied. One often demands that the superfield has to be real. The expansion of the superfield Φ in θ and $\bar{\theta}$ reads

$$\Phi(x, \theta, \bar{\theta}) = \phi(x) + \bar{\theta}\psi(x) + \bar{\psi}(x)\theta + \bar{\theta}\theta F(x). \tag{3.2}$$

The SuSy transformations for the component fields have the structure

$$\delta\phi \sim \psi, \quad \delta\psi \sim \partial\phi + F, \quad \delta F \sim \partial\psi. \tag{3.3}$$

The superfield has too many degrees of freedom. In order to reduce these, the superfield has to obey constraints that are compatible with supersymmetry, i. e. they have to anticommute with the supercharges. The supercharges D and $\bar{D}$ fulfil these requirements so they are used to put constraints on the superfield.

In superspace it is straightforward to construct supersymmetric actions. For this it is needed that the product of superfields is again a superfield and that the $\bar{\theta}\theta$-component of the superfield transforms into a total derivative under SuSy transformations. Therefore the highest component of any analytic function of superfields and its super-covariant derivatives yield a function of the component fields that changes by a total derivative under SuSy transformations and gives the Lagrange density.

To obtain the component formulation, the superspace integral has to be performed, that is the Grassmann coordinates in the action have to be integrated out. Due to the properties of the Grassmann numbers this projects onto the highest component. From the action of the supersymmetry generators on the superfield the supersymmetry transformations of component fields can be read off after an expansion in the Grassmann parameters.

Although we mostly use the component formulation in the following, we need the superspace formulation for the construction of the supersymmetric cutoff action. Following this procedure it is guaranteed that the regulator does not break supersymmetry. Implementing this ansatz, the regulator structure necessary to preserve supersymmetry differs from the one usually found for theories with bosons and fermions.

3.3 Spontaneous breaking of supersymmetry

Spontaneous supersymmetry breaking means that the variation of some field under SuSy transformations does not vanish in the ground state. This implies that the auxiliary field acquires a non-vanishing vacuum expectation value as it is the only Lorentz scalar in the transformation. Equivalently, supersymmetry breaking implies that the ground state energy becomes non-zero. This follows from

$$E_{\min} \equiv \langle 0|E|0\rangle = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} |Q_i|0\rangle|^2 \tag{3.4}$$

which implies

$$\langle 0|E|0\rangle \neq 0 \Rightarrow Q_i|0\rangle \neq 0. \tag{3.5}$$

The ground state energy can also be obtained from the expectation value of the scalar potential

$$E_{\min} = \langle 0|V|0\rangle. \tag{3.6}$$

From this we conclude that SuSy is broken if and only if the minimum of the potential is positive. This criterion is also valid even if no off-shell formulation with auxiliary fields exists for a theory.

From the Goldstone theorem it follows that supersymmetry breaking yields a massless spin $^1/_2$ particle which is called *Goldstino* [99]. A proof for this can be found e. g. in [100].

There are several mechanisms that are discussed for spontaneous breaking of supersymmetry, for example the *O'Raifeartaigh mechanism* [101], a scalar model with three multiplets where the equations of motion for the auxiliary fields are such that they cannot vanish simultaneously. Other mechanisms are discussed e. g. in the papers by P. Martin [69], E. Poppitz and S. Trivedi [100] or E. Witten [102]. For a review on spontaneous SuSy breaking see for example the review by Y. Shadmi and Y. Shirman [103].

3.4 Kähler potential

In order to properly define the Kähler potential in chapter 7, we give a short introduction to complex manifolds in this section, following lecture notes by J. v. Holten [104].

On an N dimensional complex manifold there exists a finite set of local complex coordinate systems $(\bar{z}^{\underline{i}}, z^i, (\underline{i}, i) = 1, \ldots, N)$ that covers the manifold such that the transition functions between two sets of coordinate systems are holomorphic. The metric on the manifold is given through the line element

$$ds^2 = g_{i\underline{i}} d\bar{z}^{\underline{i}} dz^i. \tag{3.7}$$

A Kähler manifold is a complex manifold with the condition that the holomorphic and antiholomorphic curl of the metric vanishes:

$$g_{i\underline{i},j} = g_{j\underline{i},i}, \quad g_{i\underline{i},\underline{j}} = g_{i\underline{j},\underline{i}} \tag{3.8}$$

From this it follows that the metric can be derived from a real function $K(z, \bar{z})$ through

$$g_{i\underline{i}}(z, \bar{z}) = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\underline{i}}} \tag{3.9}$$

with $K(z, \bar{z})$ being the Kähler potential.

A supersymmetric theory with a (complex) chiral superfield, such as the $\mathcal{N} = 2$ Wess-Zumino model in two dimensions, can be constructed from a real superfield-valued Kähler potential. The Kähler potential also allows for an elegant way to construct (supersymmetric) non-linear sigma models, see e. g. the textbook by P. West [93].

4 Supersymmetric quantum mechanics

In order to extend the FRG to supersymmetric theories the formalism is first applied to supersymmetric quantum mechanics (SuSy-QM). This model was initiated by H. Nicolai [105] in 1976 and later formulated by E. Witten [102] in 1981. Witten suggested this model in order to understand supersymmetry breaking in a simple non-relativistic system rather than in the complicated setting of supersymmetric gauge theories. This model turned out to be interesting in its own right, see e. g. the articles by A. Wipf [94] and A. Kahre [106] or the report by F. Cooper, A. Kahre and U. Sukhatme [107].

SuSy-QM can be formulated as a $0 + 1$ dimensional field theory. As it is the simplest supersymmetric model that allows for supersymmetry breaking it is well suited to study how the FRG can be extended to supersymmetric theories. The techniques developed in this chapter can easily be generalised to Wess-Zumino models in various dimensions.

It is possible to calculate the energy of the first exited state numerically by diagonalising the Hamiltonian. This offers a benchmark test for the applicability of the FRG to supersymmetric theories. The results reported in this chapter are published in [108]. In this paper additionally a formulation of the flow equations in superspace can be found which yields the same results as the formulation in components presented here.

SuSy-QM with broken symmetry has previously been investigated with non-perturbative renormalisation methods by A. Horikoshi et. al. [109]. They found good agreements for the ground state energy and the first excited state in regions where quantum tunnelling is not important. M. Weyrauch [110] found that an inclusion of a wave-function renormalisation improves the results in this regime.

Both Horikoshi et. al. and Weyrauch used regulators which break supersymmetry explicitly. This makes it difficult to distinguish between explicit SuSy breaking by the regulator and spontaneous breaking that is inherent in the theory. For this reason we will always consider a regulator that preserves supersymmetry. The chapter focuses on unbroken SuSy.

This chapter is organised as follows: First the model is presented and the convention for the notations are described. The supersymmetric flow equations are then derived at leading and next-to-leading order and the first excited state is calculated for different regulators. The chapter concludes with a discussion of terms beyond next-to-leading order and the differences that arise in a supersymmetric theory compared to a non-supersymmetric one.

4.1 Description of the model

We consider supersymmetric quantum mechanics in the Euclidean formulation. The real superfield written in components reads

$$\Phi(x) = \phi(x) + \bar{\theta}\psi(x) + \bar{\psi}(x)\theta + \bar{\theta}\theta F(x). \tag{4.1}$$

It contains a bosonic field ϕ, an auxiliary field F and fermions $\psi, \bar{\psi}$. The θ and $\bar{\theta}$ are constant anticommuting spinors. A function of the superfield has the expansion

$$W(\Phi) = W(\phi) + (\bar{\theta}\psi + \bar{\psi}\theta)W'(\phi) + \bar{\theta}\theta(FW'(\phi) - W''(\phi)\bar{\psi}\psi) \tag{4.2}$$

with $W(\Phi)$ being polynomial in the superfield and $W(\phi)$ the same polynomial of the bosonic field. The supercharges that generate the supersymmetry transformations $\delta_\varepsilon = \bar{\varepsilon}Q - \bar{Q}\varepsilon$ are

$$Q = \mathrm{i}\partial_{\bar{\theta}} + \theta\partial_\tau \quad \text{and} \quad \bar{Q} = \mathrm{i}\partial_\theta + \bar{\theta}\partial_\tau, \tag{4.3}$$

their anticommutator is the generator of time translations

$$\{Q, \bar{Q}\} = 2\mathrm{i}\partial_\tau = 2H. \tag{4.4}$$

The variation of the superfield reads

$$\delta_\varepsilon\Phi = \bar{\varepsilon}\left(\mathrm{i}\psi + \mathrm{i}\theta F + \theta\dot{\phi} + \theta\bar{\theta}\dot{\psi}\right) - \left(\mathrm{i}\bar{\psi} + \mathrm{i}\bar{\theta}F - \bar{\theta}\dot{\phi} + \bar{\theta}\theta\dot{\bar{\psi}}\right)\varepsilon. \tag{4.5}$$

For the components this implies the transformations

$$\delta_\varepsilon\phi = \mathrm{i}\bar{\varepsilon}\psi - \mathrm{i}\bar{\psi}\varepsilon, \quad \delta_\varepsilon\psi = (\dot{\phi} - \mathrm{i}F)\bar{\varepsilon}, \quad \delta_\varepsilon\bar{\psi} = \bar{\varepsilon}(\dot{\phi} + \mathrm{i}F), \quad \delta_\varepsilon F = -\bar{\varepsilon}\dot{\psi} - \dot{\bar{\psi}}\varepsilon. \tag{4.6}$$

The covariant derivatives are $D = \mathrm{i}\partial_{\bar{\theta}} - \theta\partial_\tau$ and $\bar{D} = \mathrm{i}\partial_\theta - \bar{\theta}\partial_\tau$. They obey the anticommutation relations $\{D, D\} = \{\bar{D}, \bar{D}\} = 0$ and $\{D, \bar{D}\} = -2\mathrm{i}\partial_\tau$.

The off-shell action for supersymmetric quantum mechanics is given by

$$S_{\text{off}}[\phi, F, \psi, \bar{\psi}] = \int \mathrm{d}\tau\mathrm{d}\theta\mathrm{d}\bar{\theta}\left[\frac{1}{4}\Phi(\bar{D}D - D\bar{D})\Phi + \mathrm{i}W(\Phi)\right] \tag{4.7}$$

$$= \int \mathrm{d}\tau\left[\frac{1}{2}\dot{\phi}^2 - \mathrm{i}\bar{\psi}\dot{\psi} + \frac{1}{2}F^2 + \mathrm{i}FW'(\phi) - \mathrm{i}W''(\phi)\bar{\psi}\psi\right]. \tag{4.8}$$

After eliminating the auxiliary field F with its equation of motion $F = -\mathrm{i}W'(\phi)$ this yields the action of a supersymmetric anharmonic oscillator:

$$S_{\text{on}}[\phi,\psi,\bar{\psi}] = \int \mathrm{d}\tau \left[\frac{1}{2}\dot{\phi}^2 - i\bar{\psi}\dot{\psi} + \frac{1}{2}W'(\phi)^2 - iW''(\phi)\bar{\psi}\psi\right] \tag{4.9}$$

For the bosonic potential $V(\phi)$ follows the identity

$$V(\phi) = \frac{1}{2}W'(\phi)^2. \tag{4.10}$$

The ground state energy is given by the minimum of the effective potential. The energy gap between the ground state and the first excited state is given by the pole of the propagator in the complex plane or the exponential decay of the correlator respectively. In the truncation considered in this chapter the wave-function renormalisation is independent of the momentum. In this case the energy gap is given by the curvature at the minimum of the effective potential:

$$E_1 - E_0 = \sqrt{\frac{\mathrm{d}^2 V(\phi)}{\mathrm{d}\phi^2}}\Bigg|_{\phi=\phi_{\text{min}}} = \sqrt{W'(\phi)W'''(\phi) + W''(\phi)^2} \tag{4.11}$$

The case with a momentum-dependent wave-function will be discussed in chapter 7.

As we are interested in unbroken supersymmetry in the following, we consider only superpotentials that are of the form $W \sim O(\phi^{2n})$. This implies a vanishing ground state energy $E_0 = 0$. In this chapter the choice of the superpotential differs from the following ones on the $\mathcal{N} = 1$ Wess-Zumino models where we consider potentials that exhibit spontaneous SuSy breaking. This requires potentials of the type $W \sim O(\phi^{2n+1})$.

4.2 The supersymmetric flow equation

In this section we sketch the derivation of the supersymmetric flow equation. The regulator structure and method can be generalised to the supersymmetric models with scalar fields in different dimensions that will be considered in the following chapters.

4.2.1 The supersymmetric cutoff action

In order to preserve supersymmetry throughout the calculations we choose the cutoff action to be quadratic in the superfields and the regulator to be a function of covariant derivatives,

$$\Delta S_k[\Phi] = \frac{1}{2}\int \frac{\mathrm{d}q}{2\pi}\mathrm{d}\theta\mathrm{d}\bar{\theta}\,\Phi(-q)R_k\left(D,\bar{D}\right)\Phi(q). \tag{4.12}$$

The function R_k has to obey the general requirements for a regulator (cf. section 2.4). Using the anticommutation relations for the covariant derivatives the regulator function can be decomposed into

$$R_k\left(D,\bar{D}\right) = \mathrm{i}r_1(k,q^2) + \frac{1}{2}\left(D\bar{D} - \bar{D}D\right) r_2(k,q^2) \tag{4.13}$$

with a factor 'i' chosen for convenience such that the function r_1 matches the potential term. In components the cutoff action reads

$$\Delta S_k[\phi,F,\bar{\psi},\psi] = \frac{1}{2}\int \frac{\mathrm{d}q}{2\pi}\left[r_2(k,q^2)\left(q^2\phi^2 + 2q\bar{\psi}\psi + F^2\right) + 2\mathrm{i}r_1(k,q^2)\left(F\phi + \bar{\psi}\psi\right)\right]. \tag{4.14}$$

In the cutoff action $r_1(k,q^2)$ plays the role of a momentum-dependent mass. The term $r_2(k,q^2)$ is a modification of the kinetic term, similar to the regulators used for non-supersymmetric theories. Therefore we choose $r_2(k,q^2)\cdot q^2$ to be a typical regulator for a bosonic theory such as the regulators in equation (2.18).

Written as a matrix in the space of the component fields $(\phi,F,\psi,\bar{\psi})$ the regulator takes the form

$$R_k = \begin{pmatrix} R_k^{\mathrm{B}} & 0 \\ 0 & R_k^{\mathrm{F}} \end{pmatrix} \quad \text{with} \quad R_k^{\mathrm{B}} = \begin{pmatrix} q^2 r_2 & \mathrm{i}r_1 \\ \mathrm{i}r_1 & r_2 \end{pmatrix} \quad \text{and} \quad R_k^{\mathrm{F}} = \begin{pmatrix} 0 & qr_2 + \mathrm{i}r_1 \\ qr_2 - \mathrm{i}r_1 & 0 \end{pmatrix}. \tag{4.15}$$

Requiring supersymmetry relates the regulators for bosonic and fermionic fields and introduces additional constraints on fermionic and bosonic regulators. Note that, in order to preserve supersymmetry, the auxiliary field has to be regularised (cf. section 4.6). The regulator structure constructed here generalises to supersymmetric models with scalar fields (Wess-Zumino models) in two to four dimensions. We will discuss these models in the following chapters.

For the components the flow equation for SuSy-QM is a flow equation for fermions and bosons with the special regulator structure given in equation (4.15) that ensures supersymmetry.

4.2.2 The supercovariant derivative expansion

In order to solve the flow equation we employ a truncation that is called *supercovariant* expansion[1]. The first term in this expansion contains no covariant derivatives, but an arbitrary function of the superfield. This approximation is called the *local potential approximation*. It corresponds to considering a scale-dependent superpotential. The second term in the expansion contains the derivatives D and $\bar{D}$ and has the form

$$Z(\Phi)D\bar{D}Z(\Phi) \quad \text{or} \quad Z(\Phi)\bar{D}DZ(\Phi) \tag{4.16}$$

with $Z(\Phi)$ an arbitrary function of the superfield. This corresponds to additionally considering a scale dependent wave-function renormalisation. Because of the anticommutation relation for

[1]This terminology changes for supersymmetric theories with more than one supercharge. What remains is an expansion that corresponds to an expansion in the auxiliary field.

the supercovariant derivatives an arbitrary function $g(D\bar{D})$ in between the functions Z can be reduced to a function $g(p)\cdot D\bar{D}$. In this way it is possible to construct a momentum-dependent wave-function renormalisation (cf. chapter 7). The third term is of the form

$$\Upsilon(\Phi)D\bar{D}\Upsilon(\Phi)D\bar{D}\Upsilon(\Phi). \tag{4.17}$$

Again an arbitrary function of $D\bar{D}$ reduces to a multiplicative function depending on the momentum.

It is important to keep in mind that this expansion is *not* an expansion in momenta as it is normally considered in the derivative expansion. It is rather an expansion in powers of the auxiliary field F: The local potential approximation contains a term linear in F, the wave function renormalisation a term proportional to F^2 and the third term is proportional to F^3. This fact is used to project out the different parts of the expansion.

4.3 Local potential approximation

We first consider the local potential approximation. In this truncation the effective action reads

$$\begin{aligned}\Gamma_k[\phi,F,\bar{\psi},\psi] &= \int \mathrm{d}\theta\mathrm{d}\bar{\theta}\int \mathrm{d}\tau\left[\frac{1}{4}\Phi(D\bar{D}-\bar{D}D)\Phi+\mathrm{i}\cdot W_k(\Phi)\right]\\ &= \int \mathrm{d}\tau\left[\frac{1}{2}\dot{\phi}^2-\mathrm{i}\bar{\psi}\dot{\psi}+\frac{1}{2}F^2+\mathrm{i}FW_k'(\phi)-\mathrm{i}W_k''(\phi)\bar{\psi}\psi\right].\end{aligned}$$

In the local potential approximation the classical action with a scale-dependent superpotential is considered. After performing the functional derivatives, the fields are assumed to be constant. In momentum space the second derivatives read

$$\begin{aligned}&\left(\Gamma_k^{(2)}+R_k\right)(q,q')\\ &=\begin{pmatrix} q^2(1+r_2)+W_k'''F+\mathrm{i}W_k^{(4)} & \mathrm{i}(W_k''+r_1) & \mathrm{i}W_k'''\bar{\psi} & -\mathrm{i}W_k'''\psi\\ \mathrm{i}(W_k''+r_1) & 1+r_2 & 0 & 0\\ -\mathrm{i}W_k'''\bar{\psi} & 0 & 0 & q(1+r_2)-\mathrm{i}(W_k''+r_1)\\ \mathrm{i}W_k'''\psi & 0 & q(1+r_2)+\mathrm{i}(W_k''+r_1) & 0\end{pmatrix}\delta(q,q').\end{aligned} \tag{4.18}$$

To calculate the right hand side of

$$\partial_k\Gamma_k=\frac{1}{2}\,\mathrm{STr}\left\{\left[\Gamma_k^{(2)}+R_k\right]^{-1}\partial_kR_k\right\}=\frac{1}{2}\,\mathrm{Tr}\,(G_k\partial_kR_k)_{BB}-\frac{1}{2}\,\mathrm{Tr}\,(G_k\partial_kR_k)_{FF}\,, \tag{4.19}$$

we need the inverse of the supermatrix[2] (4.18). For this the inverse propagator is decomposed into bosonic and fermionic parts:

$$\Gamma_k^{(2)} + R_k \equiv G_k^{-1} = G_{0,k}^{-1} + \bar{\psi} M_{1,k} + M_{2,k}\psi + \bar{\psi} M_{3,k}\psi \tag{4.20}$$

The propagator itself reads

$$G_k = G_{0,k} - G_{0,k}(\bar{\psi} M_1 + M_2\psi) G_{0,k} + G_{0,k}\,(M_1 G_{0,k} M_2 - M_2 G_{0,k} M_1 - M_3)\, G_{0,k}\bar{\psi}\psi\,. \tag{4.21}$$

The explicit form of the matrices and the inversion of the propagator can be found in appendix B.1. Projecting on the terms linear in the auxiliary fields[3] leads to the following flow equation[4]:

$$\partial_k W_k' = -\frac{W_k'''}{2}\int \frac{\mathrm{d}q}{2\pi}\,\frac{\partial_k r_2\left[(1+r_2)^2 p^2 - (W_k''+r_1)^2\right] + 2(1+r_2)\partial_k r_1 (W_k''+r_1)}{\left[(1+r_2)^2 p^2 + (W_k''+r_1)^2\right]^2}. \tag{4.22}$$

Integrating with respect to ϕ (and dropping the irrelevant constant of integration) finally yields the flow equation for the superpotential

$$\partial_k W_k(\phi) = \frac{1}{2}\int \frac{\mathrm{d}q}{2\pi}\,\frac{(1+r_2)\partial_k r_1 - \partial_k r_2 (W_k''(\phi)+r_1)}{(1+r_2)^2 p^2 + (W_k''(\phi)+r_1)^2}. \tag{4.23}$$

As it is required by supersymmetry the flow equation for the superpotential coincides with the one obtained by a projection on the terms proportional to $\bar{\psi}\psi$. This projection yields an equation for $W_k''(\phi)$. Details on the calculation can be found in appendix B.2. There the equality of both equations is explicitly shown.

4.3.1 Discussion of different regulators

In this section we discuss and compare different regulators or regularisation schemes by varying the regulator.

As supersymmetric quantum mechanics is an ultraviolet finite theory we can use very simple regulators. However, these regulators will not be sufficient in the more complicated models as we shall see in the following chapters.

In the following we will focus on the simplest nontrivial potential given by[5]

$$W_{\mathrm{cl}}(\phi) = e\phi + \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3 + \frac{a}{4}\phi^4. \tag{4.24}$$

[2] For the inversion of a supermatrix see e. g. [111].

[3] This projection can be achieved in general by a functional derivative with respect to the auxiliary field and then setting the auxiliary field and the fermions equal to zero.

[4] For the explicit calculation see appendix B.2.

[5] A short remark concerning the dimensions of the couplings: For a numerical treatment the couplings have to be dimensionless. In this chapter all couplings and the fields are measured in units of the mass. This implies that the mass parameter m is identical to one throughout this chapter.

The Callan-Symanzik regulator

Due to the ultraviolet finiteness of SuSy-QM only an infrared regulator is needed. For this reason, the simplest choice, $r_2 = 0$ and $r_1 = k$, is sufficient. We refer to this as the *Callan-Symanzik regulator* in the following as it is very similar to the one used in the Callan-Symanzik equation. For this regulator equation (4.23) simplifies to

$$\partial_k W_k(\phi) = \frac{1}{4} \cdot \frac{1}{k + W''(\phi)}. \tag{4.25}$$

We compare the polynomial approximation and the numerical solution of the partial differential equation.

As a benchmark test for the quality of the approximations E_1, the energy of the first excited state, is determined from the curvature of the effective potential at its minimum ϕ_{min}. The effective potential V_k is defined as $\lim_{k\to 0} \frac{1}{2}(W_k')^2$. From $W'_{k\to 0}(\phi_{\text{min}}) = 0$ we obtain the energy of the first excited state as

$$E_1 = W''_{k\to 0}(\phi_{\text{min}}). \tag{4.26}$$

For a polynomial expansion of the superpotential the ansatz reads

$$W_k(\phi) = \sum_n \frac{a_n(k)}{n}\phi^n \quad \text{with} \quad W_{k\to\Lambda} = W_{\text{cl}} = e\phi + \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3 + \frac{a}{4}\phi^4. \tag{4.27}$$

Since only $W_k''(\phi)$ enters on the right hand side of the flow equation, the couplings a_0 and a_1 will not determine the flow of the other couplings. This generalises to Wess-Zumino models in various dimensions as well. As long as the superpotential is convex it can always be expanded around $\phi = 0$. At the cutoff Λ the non-vanishing coupling constants are $(a_1, a_2, a_3, a_4) = (e, m, g, a)$. For the ansatz given in equation 4.27 the classical superpotential becomes non-convex if $g^2 > 3ma$. An expansion around the fields minimising $W_k''(\phi)$ would be better adjusted to the flow as this has the largest contributions to the flow. In this case the expansion reads

$$W_k(\phi) = \sum_n \frac{\tilde{a}_n(k)}{n}(\phi - \phi_0(k))^n, \quad W_k'''(\phi_0) = 2\tilde{a}_3 = 0. \tag{4.28}$$

In general, the field ϕ_0 minimising the superpotential does not coincide with the field ϕ_{min} minimising the bosonic potential. The system of coupled ordinary differential equations for the coupling constants can be derived by comparison of coefficients.

As an even potential remains even during the flow, all odd couplings obey $\tilde{a}_{2n+1}(k) = 0$ for $n > 1$. Moreover we have $\partial_k \phi_0(k) = 0$, implying that the minimum of W_k'' is scale invariant. The system of differential equations is given in [108] up to order $N = 10$.

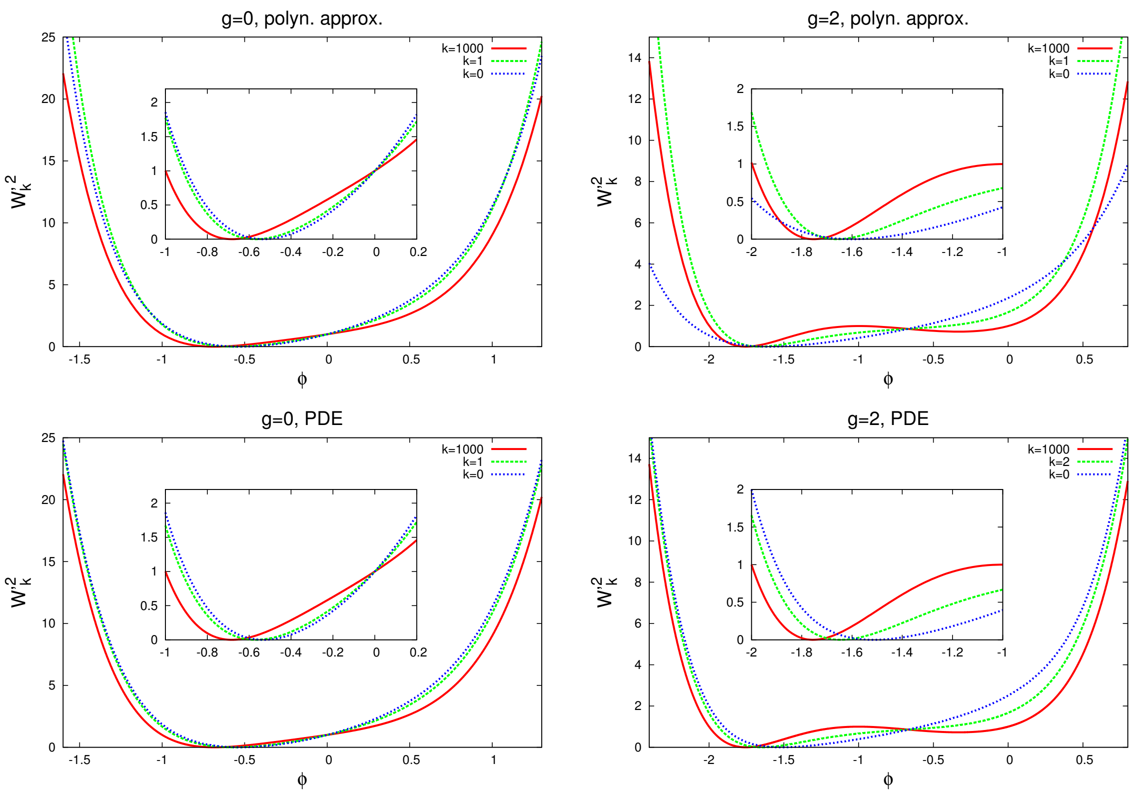


Figure 4.1: Flow of $W'^2(\phi)$ with Callan-Symanzik regulator for $W'_{\text{cl}} = 1 + \phi + g\phi^2 + \phi^3$.
Left panels: $g = 0$, *Right panels*: $g = 2$, *First row*: polynomial approximation to order ϕ^{10}, *Second row*: solution of the partial differential equation (4.25).

	g	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
pol. expan.	ϕ^4	2.008	1.960	1.895	1.815	1.722	1.615	1.497	1.371	1.237	1.097
	ϕ^6	2.205	2.140	2.064	1.980	1.889	1.794	1.699	1.608	1.530	1.472
	ϕ^8	2.214	2.146	2.070	1.987	1.898	1.808	1.721	1.646	1.596	1.590
	ϕ^{10}	2.201	2.135	2.060	1.977	1.888	1.798	1.711	1.638	1.595	1.612
PDE	CS	2.203	2.137	2.062	1.979	1.890	1.798	1.710	1.633	1.584	1.590
	exp	2.195	2.130	2.055	1.972	1.884	1.791	1.701	1.622	1.569	1.684
	θ	2.197	2.132	2.058	1.975	1.888	1.794	1.705	1.626	1.576	1.581
	exact	2.022	1.970	1.905	1.827	1.738	1.639	1.534	1.426	1.323	1.235

Table 4.1: *Upper part*: Energy of the first excited state calculated in different orders of the polynomial approximation with the Callan-Symanzik regulator for $e = m = a = 1$. *Lower part:* Solutions from the partial differential equation (PDE) for the Callan-Symanzik (CS), the exponential (exp) and the θ-regulator. For comparison, also the exact values from a numerical diagonalisation of the Hamiltonian are given.

As the superpotential becomes non-convex, $\phi_{\min}$ moves away from the expansion point ϕ_0 signalling the breakdown of the polynomial approximation for large couplings. It is known from non-supersymmetric quantum mechanics that for non-convex potentials the polynomial approximation fails and the full partial differential equation has to be solved [109, 112, 113, 114].

We solve the partial differential equation with the `NDSolve` routine of Mathematica. ϕ is chosen in the range $(-200, 200)$, on the boundary the potential is kept at its classical value. The flow of $W_k'^2$ from the polynomial approximation and the solution to the partial differential equation is depicted in figure 4.1.

In table 4.1 the results for the energy E_1 from polynomial approximations to different orders and the solution of the partial differential equation are listed. For convex superpotentials the results obtained from the former converge rapidly to the ones from the latter. The results deviate about 10% from the exact results. For non-convex potentials the results deviate even more. In the next paragraph two different regulators are investigated. It will turn out that the observed large deviation is a problem of the LPA in a supersymmetric theory and not of this particular regulator.

Exponential- and θ-regulator

The Callan-Symanzik regulator serves only as an infrared regulator. The exponential (left) and a θ-regulator, defined as

$$r_1^{(\exp)}(q^2, k) = k\, \mathrm{e}^{-q^2/k^2}, \quad r_1^{(\theta)}(q^2, k) \;\; = \sqrt{k^2 - q^2}\,\theta(k^2 - q^2) \tag{4.29}$$

serve as infrared *and* ultraviolet regulators. The corresponding flow equations read

$$\begin{aligned} \partial_k W_k^{(\exp)}(\phi) &= \frac{1}{2k^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{2\pi} \frac{(k^2 + 2q^2) e^{-q^2/k^2}}{q^2 + (W_k''(\phi) + k e^{-q^2/k^2})^2} \quad \text{and} \\ \partial_k W_k^{(\theta)}(\phi) &= \frac{1}{4\pi} \frac{k}{|k^2 - W_k''^2|} \left(\pi \left(1 - \operatorname{sign} W_k''\right) + 2 \arctan \frac{|k^2 - W_k''^2|}{2k W_k''} \right). \end{aligned} \tag{4.30}$$

The results in the convex regime do not deviate much from each other (see table 4.1). In the non-convex regime they do not yield better results than the Callan-Symanzik regulator. The ground state energy from the LPA has an error of about 10% for the choice of couplings $(e, m, a) = (1, 1, 1)$. This is due to contributions from higher orders in the auxiliary fields that are neglected in the LPA. Recall that the effective action is expanded in the supercovariant derivatives and the auxiliary field mixes different orders of the momentum. Because of the auxiliary fields, wave function renormalisation has contributions to zeroth order of the momentum.

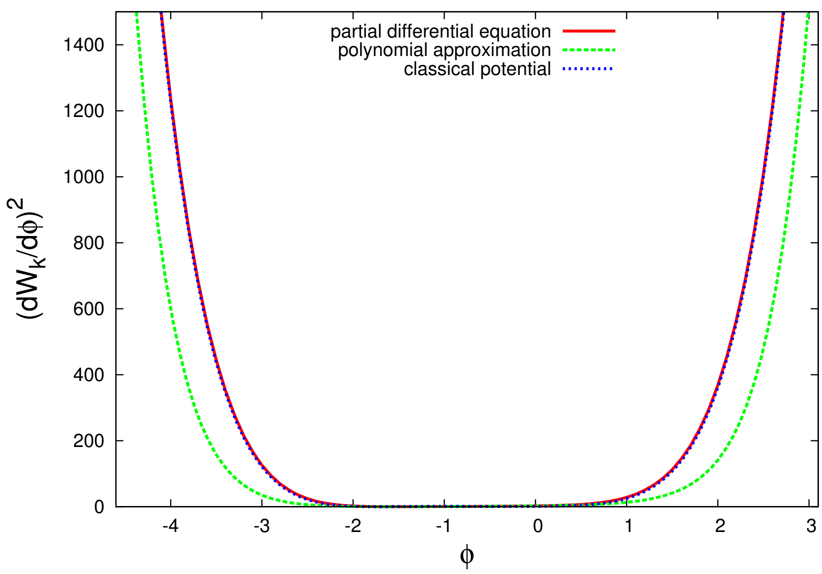


Figure 4.2: The effective potential V_k calculated with the Callan-Symanzik regulator for a non-convex classical superpotential. The polynomial approximation does not reproduce the global structure which should not deviate much from the classical potential. The solution of the partial differential equation gives the correct asymptotic behaviour. The parameters in the classical potential are $e = m = a = 1$ and $g = 2$.

4.3.2 The global structure of the potential

For a convex superpotential at the cutoff scale the polynomial approximation works quite well near the origin but it breaks down in the non-convex case. In figure 4.2 the asymptotic behaviour of the effective potential is shown, calculated from the solution of the partial differential equation and from the polynomial approximation. As expected, the polynomial approximation fails to reproduce the correct asymptotic behaviour whereas the full partial differential equation succeeds.

4.4 Next-to-leading order approximation

In this section we investigate how the results of the previous section change if a wave-function renormalisation is included. A constant wave-function renormalisation is driven by the odd couplings and is scale invariant for this model. Therefore we have to consider a field-dependent wave-function renormalisation. The ansatz for the effective action in this case reads[6]

$$\begin{aligned}\Gamma_k[\phi, F, \bar{\psi}, \psi] &= \int \mathrm{d}\tau \mathrm{d}\theta \mathrm{d}\bar{\theta} \left[\frac{1}{4} Z_k(\Phi)(D\bar{D} - \bar{D}D) Z_k(\Phi) + \mathrm{i} W_k(\Phi) \right] \\ &= \int \mathrm{d}\tau \left[\frac{Z_k'(\phi)^2}{2} \left(\dot{\phi}^2 - 2\mathrm{i}\bar{\psi}\dot{\psi} + F^2 \right) - Z_k'(\phi) Z_k''(\phi) \left(\mathrm{i}\dot{\phi} + F \right) \bar{\psi}\psi + \mathrm{i} F W_k'(\phi) - \mathrm{i} W_k''(\phi) \bar{\psi}\psi \right] \end{aligned} \tag{4.31}$$

[6] In order to include a field-dependent wave-function renormalisation such that supersymmetry is preserved, two functions are necessary on which the supercovariant derivatives act. For this reason Z_k^2 instead of Z_k is considered in the following. This is different to the terminology normally used in theories without supersymmetry.

where primes denote derivatives with respect ϕ. For $Z(\Phi) = \Phi$ the truncation in the previous section is recovered.

For Wess-Zumino models with spontaneous supersymmetry breaking a constant wave-function renormalisation has a nontrivial flow equation and then it is sufficient to take $Z_k(\Phi) = Z_k \cdot \Phi$. However, for the supersymmetric quantum mechanics the full ϕ-dependence is needed.

In order to respect reparameterisation invariance of the physical quantities [17] under rescaling we choose the cutoff action to be

$$\begin{aligned}\Delta S_k &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}\theta \mathrm{d}\bar{\theta}\, Z_k'(\bar{\Phi})^2 \Phi \left[\mathrm{i}r_1 + \frac{r_2}{2}(D\bar{D} - \bar{D}D)\right]\Phi \\ &= \int \mathrm{d}q\, Z_k'(\bar{\phi})^2 \left(\frac{1}{2}q^2 r_2 \phi^2 + \frac{r_2}{2}F^2 + \mathrm{i}r_1 F\phi + (qr_2 - \mathrm{i}r_1)\bar{\psi}\psi\right)\end{aligned} \tag{4.32}$$

with $\bar{\Phi} = (\bar{\phi}, 0, 0, 0)$ being a background field. This ansatz of a spectrally adjusted flow [78, 115] is inspired by functional optimisation [22]. The field $\bar{\phi}$ can be understood as a parameter labelling the classes of regulators.

The flow of Z_k can either be read off from $\dot{\phi}^2$, F^2 or $\bar{\psi}\dot{\psi}$. The simplest choice is F^2 because there are no time derivatives involved. However, due to this mixing of powers of the momentum, the wave-function renormalisation has a strong influence on the flow of the superpotential. This explains the large error in the ground state energy found in the LPA compared to the exact result. It is related to the fact that the F^2-term in the off-shell formulation originates from the kinetic term but in the process of integrating out the auxiliary field enters the definition of the effective bosonic potential. Projecting on vanishing $\bar{\psi}\psi$ as well as on constant scalar field and considering the Callan-Symanzik regulator leads to the coupled flow equations

$$\begin{aligned}\partial_k W_k'(\phi) &= -W_k''' \frac{\mathcal{N}}{4\mathcal{D}^2}, \\ Z_k'(\phi)\partial_k Z_k'(\phi) &= \left(\frac{4Z_k''(\phi)W_k'''(\phi)}{\mathcal{D}} - \left(Z_k''(\phi)Z_k'(\phi)\right)' - \frac{3Z_k'(\phi)^2 W_k'''(\phi)^2}{4\mathcal{D}^2}\right)\frac{\mathcal{N}}{4\mathcal{D}^2},\end{aligned} \tag{4.33}$$

with

$$\mathcal{N} = (1 + k\partial_k)Z_k'\left(\bar{\phi}\right)^2 \quad \text{and} \quad \mathcal{D} = W''(\phi) + kZ_k'\left(\bar{\phi}\right)^2. \tag{4.34}$$

To solve these equations a value for the background field $\bar{\phi}$ has to be chosen. A good choice would be $\bar{\phi} = \phi_{\mathrm{min}}$ because E_1 is of interest, but this value has to be determined through a self-consistency calculation. From gauge theory it is known that it suffices to take $\bar{\phi} = \phi$, see e. g. [18]. This already improves the results tremendously. For this reason we will use this approximation in the following. The flow equation then simplifies a great deal.

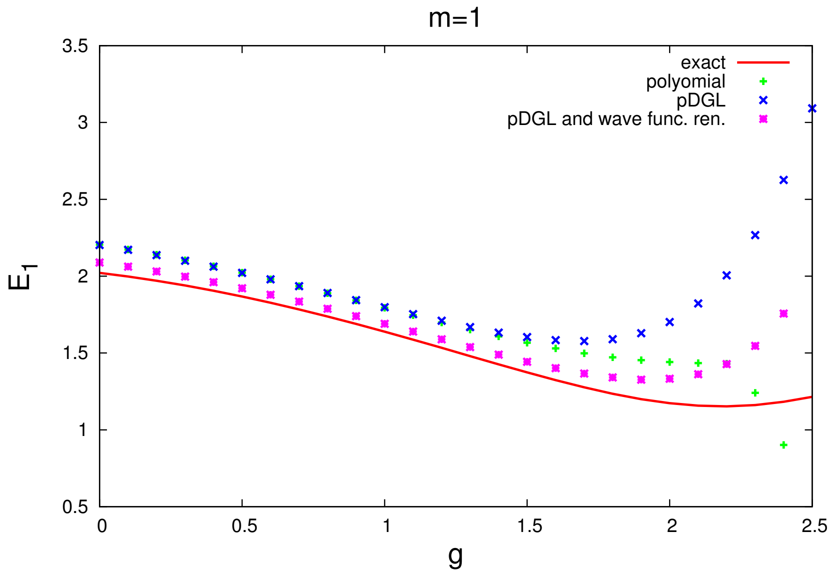


Figure 4.3: Dependence of the energy on the coupling g for $(e, m, a) = (1, 1, 1)$. The polynomial approximation is of order $n = 10$.

g	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
PDE	2.203	2.137	2.062	1.979	1.890	1.798	1.710	1.633	1.584	1.590
PDE+WF	2.089	2.031	1.961	1.879	1.788	1.690	1.589	1.489	1.402	1.341
exact	2.022	1.970	1.905	1.827	1.738	1.639	1.534	1.426	1.323	1.235

Table 4.2: Energy of the first excited state for the classical superpotential (4.27) with $(e, m, a) = (1, 1, 1)$ and various values of g calculated from the solution to flow equations with Callan-Symanzik regulator with and without wave-function renormalization.

With wave-function renormalisation included the on-shell effective bosonic potential is given by

$$V(\phi) = \frac{1}{2}\left(\frac{W'_{k\to 0}(\phi)}{Z'_{k\to 0}(\phi)}\right)^2. \tag{4.35}$$

The curvature of this potential with respect to canonically renormalised fluctuations $\chi = Z(\phi)$ yields the energy gap. The energy of the first excited state reads

$$E_1 = \sqrt{\frac{d^2 V\left(Z^{-1}(\chi)\right)}{d\chi^2}}\Bigg|_{\chi_{\min}=Z(\phi_{\min})} = \lim_{k\to 0} \frac{W''_k(\phi)}{\left(Z'_k(\phi)\right)^2}\Bigg|_{\phi=\phi_{\min}}. \tag{4.36}$$

Note that there are no additional terms from differentiation of the wave-function renormalisation because $W'(\phi)|_{\phi=\phi_{\min}} = 0$. In table 4.2 we list the results with and without wave-function renormalisation .

The wave-function renormalisation improves the results for $(e, m, a) = (1, 1, 1)$ considerably. This is shown in figure 4.3.

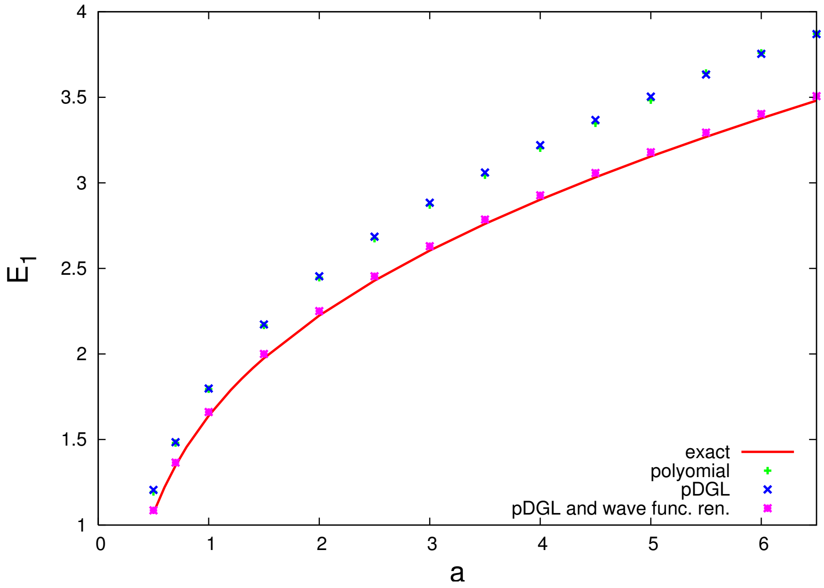


Figure 4.4: Dependence of the energy on the coupling a, $e = m = g = 1$. The polynomial approximation is of order $n = 10$

If the classical potential is in the convex regime of the superpotential, i. e. $g < \sqrt{3a}$ the results for the energy gap are independent of the regulator and an accuracy of up to 1% is achieved if the wave-function renormalisation is included in the truncation. As soon as the superpotential enters the non-convex regime, the results obtained with the flow equation deviate strongly from the exact solutions. We expect that this is due to terms of higher order in the derivative expansion, such as $\Phi[D\bar{D}\Phi]^2 \sim F^3 + \dots$ (cf. section 4.5). As the expansion in super-covariant derivatives mixes different orders of momentum, these can influence the flow equations at lower order. However, as can be seen from a diagrammatic expansion of the flow equation (cf. appendix F), auxiliary field operators (and their SuSy partners) that come with powers larger than F^3 do not directly contribute to the flow equation of the superpotential. Because of this it is reasonable to expect a good convergence at next-to-next-to-leading order.

Up to now we only considered a dependence of the energy of the first excited state on the coupling g. In order to study the dependence on the coupling a we choose the other parameters to be $e = m = g = 1$ such that the superpotential at the ultraviolet cutoff is convex. Even for large couplings a the results with wave-function renormalisation reproduce the exact results up to a 1% accuracy. This is shown in figure 4.4.

4.5 Beyond next-to-leading order

A truncation beyond next-to-leading order corresponds to taking into account both a term cubic in the auxiliary field and its supersymmetric partner terms. This is obtained from

$\Phi\left[\frac{1}{2}\left(D\bar{D}-\bar{D}D\right)\Phi\right]^2$. To calculate this term in components, we first consider

$$\frac{1}{2}\left(D\bar{D}-\bar{D}D\right)\Phi=\left(F-\mathrm{i}\dot{\phi}\right)-\frac{\mathrm{i}}{2}\left(\dot{\bar{\psi}}\theta-\bar{\theta}\dot{\psi}\right)-\bar{\theta}\theta\left(\mathrm{i}\dot{F}+\ddot{\phi}\right). \tag{4.37}$$

Taking the square of this expression yields

$$\left(\frac{1}{2}\left(D\bar{D}-\bar{D}D\right)\Phi\right)^2=\left(F-i\dot{\phi}\right)^2-2\mathrm{i}\left(\dot{\bar{\psi}}\theta-\bar{\theta}\dot{\psi}\right)\left(F-i\dot{\phi}\right)-2\bar{\theta}\theta\left(i\dot{F}+\ddot{\phi}\right)\left(F-i\dot{\phi}\right)+\frac{1}{2}\bar{\theta}\theta\dot{\bar{\psi}}\dot{\psi}. \tag{4.38}$$

For the $\bar{\theta}\theta$ component we finally obtain

$$\begin{aligned}\Phi\left(D\bar{D}\Phi\right)^2\Big|_{\bar{\theta}\theta}=&F\left(F-\mathrm{i}\dot{\phi}\right)^2+2\mathrm{i}\left(\dot{\bar{\psi}}\psi+\bar{\psi}\dot{\psi}\right)\left(F-\mathrm{i}\dot{\phi}\right)-2\phi\left(\mathrm{i}\dot{F}+\ddot{\phi}\right)\left(F-\mathrm{i}\dot{\phi}\right)+\frac{1}{2}\phi\dot{\bar{\psi}}\dot{\psi}\\ =&F^3-\mathrm{i}F^2\dot{\phi}+\dot{\phi}^2F+2\mathrm{i}\left(\dot{\bar{\psi}}\psi+\bar{\psi}\dot{\psi}\right)\left(F-\mathrm{i}\dot{\phi}\right)+2\mathrm{i}\phi\dot{\phi}\ddot{\phi}+\frac{1}{2}\phi\dot{\bar{\psi}}\dot{\psi}\end{aligned} \tag{4.39}$$

In matrix notation the second functional derivatives of this term reads

$$\Delta_k^{(2)}(q,q')=\begin{pmatrix}4q'q^2\phi+2qq'F & -2q'F & 2qq'\bar{\psi} & -2qq'\psi\\ 2q'F & 3F & 2q\bar{\psi} & -2q\psi\\ 2qq'\bar{\psi} & 2q\bar{\psi} & 0 & 2q'F-2qF\\ -2qq'\psi & -2q\bar{\psi} & -2q'F+2qF & 0\end{pmatrix}\delta(-q-q'). \tag{4.40}$$

In expression (4.40) the fields are constant. Beyond next-to-leading order the auxiliary field enters in the fermionic propagator. Thus, the fermionic propagator also influences the flow equation for the bosonic superpotential which is not the case at next-to-leading order.

4.6 Differences between theories with and without supersymmetry

Before concluding this chapter, we discuss of the differences in the flow equations of theories with and without supersymmetry.

In order to illustrate the differences, we consider a purely bosonic theory with an auxiliary field. The auxiliary field is introduced similar as in the supersymmetric case. Here the action reads:

$$S=\int \mathrm{d}^d x\left(\frac{1}{2}\dot{\phi}^2-\frac{1}{2}F^2+W'(\phi)F\right) \tag{4.41}$$

Note that this is a truncation in terms of the auxiliary field F as well as in the scalar field ϕ. In the full effective action a potential for the auxiliary field has to be taken into account.

The equation of motion of the auxiliary field is $F = W'(\phi)$ and the bosonic potential is defined as $V(\phi) = \frac{1}{2}W'(\phi)^2$. In this case it is natural to use a regulator for the bosonic field

$R_k(q^2) = q^2 r_2(k, q^2)$ and none for the auxiliary field. The flow equation for the potential is determined by projecting on constant fields which yields

$$\partial_k \left(\frac{1}{2} F^2 + W'(\phi) F \right) = \frac{1}{2} \operatorname{Tr} \left[\frac{q^2 \partial_k r_2}{(r_2 + 1) q^2 + W''(\phi)^2 + F W'''(\phi)} \right]. \tag{4.42}$$

Using the equations of motion for the auxiliary field, this gives the well-known flow equation for the bosonic potential

$$\partial_k V_k(\phi) = \frac{1}{2} \operatorname{Tr} \left[\frac{q^2 \partial_k r_2}{(r_2 + 1) q^2 + V_k''(\phi)} \right]. \tag{4.43}$$

In the supersymmetric case the auxiliary field has to be regularised as well, or else supersymmetry is broken. Due to this regularisation, the auxiliary field becomes dynamic and the bosonic potential is not directly accessible. The quantity that is of interest now is the flow of the superpotential from which the effective bosonic potential is calculated in the end.

The truncation in superspace enforces a scale-dependent superpotential. Its flow equation can be derived in component formulation through the projection on vanishing auxiliary field or it can be calculated directly in superspace, which yields the same result.

Consider a similar procedure in the purely bosonic theory above. The results for the potential V calculated from the truncation

$$\Gamma_k = \int \frac{1}{2} \dot{\phi}^2 + V_k(\phi) \quad \text{and from} \quad \Gamma_k = \int \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} F^2 + W_k'(\phi) F \tag{4.44}$$

differ. This is caused by the different truncations because different types of diagrams are resummed. Additionally, the action

$$\Gamma_k = \int \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} F^2 + F W_k'(\phi) \tag{4.45}$$

is a different truncation in terms of ϕ than

$$\Gamma_k = \int \frac{1}{2} \dot{\phi}^2 + V_k(\phi). \tag{4.46}$$

Because the auxiliary field introduces terms at zero momentum, for a consistent derivative expansion[7] a potential for the auxiliary field has to be taken into account. Such a potential for the auxiliary field modifies the relation between $W_k'(\phi)$ and F and therefore the effective potential $V_k(\phi)$ calculated from integrating out the auxiliary fields. After integrating out the auxiliary fields, the same diagrams as in the description in terms of ϕ can only be obtained if a full auxiliary field potential is included.

[7] *not* a supercovariant derivative expansion

In a supersymmetric theory, however, including a potential for the auxiliary field introduces additional terms with derivatives due to the superpartners. This leads to additional differences in the flow equations of supersymmetric and non-supersymmetric theories.

4.7 Lessons to be learnt from SuSy-QM

In this chapter we have successfully demonstrated that it is possible to extend the FRG to a supersymmetric model. The main ingredient is the choice of a supersymmetric regulator function quadratic in the superfields. This implies that the regulators for fermions and bosons have to obey certain relations which ensure that supersymmetry is preserved.

In order to solve the flow equation non-perturbatively we employ an expansion of the effective action in super-covariant derivatives. In this expansion terms without time derivatives appear even at higher orders of super-covariant derivatives. This is due to the presence of the auxiliary field and makes it necessary to go to next-to-leading order in order to obtain quantitatively correct results even for small couplings. This implies that the higher powers in the auxiliary field, or more general, an auxiliary field potential, have a strong influence when the coupling constants become large. This result is surprising because the anomalous dimension is still small. In non-supersymmetric theories this is a signal that the next-to-leading order represents already a quite reasonable truncation.

The reason for the mixing of different orders of the momentum lies in the nature of supersymmetry. In the off-shell formulation of a theory with a scalar multiplet, the auxiliary field and the derivative of the scalar field occur on equal footing. This can easily be seen from the SuSy transformation of the fermionic field ψ which is proportional to $\dot{\phi} - \mathrm{i}F$.

The physical order parameter that was investigated, the energy of the first excited state, should be a universal quantity and therefore not depend on the regulator. We have demonstrated that this is indeed the case by considering three different regulators.

We will employ the techniques developed in this chapter to study the two-dimensional $\mathcal{N} = 1$ Wess-Zumino model in the next chapter. SuSy-QM can be derived from the two-dimensional model by dimensional reduction. Because of this, it is no surprise that the regulator and the structure of the flow equation carry over to this two-dimensional model as well.

5 The two-dimensional $\mathcal{N} = 1$ Wess-Zumino model

This chapter extends and generalises the results obtained within supersymmetric quantum mechanics to the Wess-Zumino model with $\mathcal{N} = 1$ in two dimensions. The Wess-Zumino model is a simple scalar model that exhibits spontaneous supersymmetry breaking. It was again first introduced and examined by E. Witten [116].

Previously it has been investigated with lattice methods. Ranft and Schiller [117] did pioneering work based on Hamiltonian Monte-Carlo methods. They found that this model exhibits a SuSy phase transition. Beccaria and co-workers [118, 119] investigated the phase diagram and the ground state energy with similar methods. Catterall and Karamov [120] investigated the phase diagram as well. A review about supersymmetry on the lattice is given by J. Giedt [121]. The investigation of this model with lattice methods represents a great challenge although there exist formulations of the lattice action that restore supersymmetry in the continuum limit as proposed by Golterman and Petcher [122]. However, the sign of the Pfaffian changes which is a potential problem for the Monte Carlo simulations. Nonetheless, recently considerable progress in treating this model on the lattice has been made [16].

The FRG does not have this problem as no Pfaffian has to be calculated. Thus, it is in principle possible to derive results that go beyond the present lattice calculations. Nevertheless all quantities considered in the following are still cutoff dependent. To remove the ultraviolet cutoff and to work with cutoff independent quantities remains for future work. Such work could be inspired by recent research on the lattice [16].

The results reported in this chapter are published in [123, 124] as well as in the proceedings [125]. This chapter is organised as follows: First the model is presented and the SuSy flow equations are derived. They are discussed in leading and next-to-leading order, for which the fixed-point structure is investigated. A model with perturbations to the Gaußian fixed point is then explored. In the end the phase transition between broken and unbroken supersymmetry is discussed as well as the behaviour of the mass as the RG scale is lowered to the infrared.

5.1 The Wess-Zumino model

The two-dimensional $\mathcal{N} = 1$ Wess-Zumino model is a supersymmetric model with one supercharge and Yukawa-like interactions. In an off-shell formulation the theory has a scalar field ϕ, a Majorana spinor field ψ and an auxiliary field F. They are combined into a real superfield

$$\Phi(x, \theta) = \phi(x) + \bar{\theta}\gamma_*\psi(x) + \frac{1}{2}(\bar{\theta}\gamma_*\theta)F(x). \tag{5.1}$$

with the constant Majorana spinor θ and with $\gamma_* = \mathrm{i}\gamma_0\gamma_1$. In the following the γ-matrices are taken to be in the Majorana representation. The spinors ψ and $\bar{\psi}$ are related by

$$\bar{\psi} = \psi^T C \tag{5.2}$$

with $C = \mathbb{1}$ the charge conjugation matrix (cf. appendix A for details on the Clifford algebra). Majorana spinors in a Majorana representation are real. The supercharges read

$$Q = -\mathrm{i}\frac{\partial}{\partial\bar{\theta}} - \partial\!\!\!/\theta, \quad \bar{Q} = -\mathrm{i}\frac{\partial}{\partial\theta} - \bar{\theta}\partial\!\!\!/, \quad \{Q, \bar{Q}\} = 2\mathrm{i}\partial\!\!\!/ \tag{5.3}$$

and from $\delta\Phi = \mathrm{i}\bar{\varepsilon}[Q, \Phi]$ the SuSy transformations are obtained. In components they read

$$\delta\phi = \bar{\varepsilon}\gamma_*\psi, \quad \delta\psi = (F + \mathrm{i}\gamma_*\partial\!\!\!/\phi)\varepsilon, \quad \delta\bar{\psi} = \bar{\varepsilon}(F - \mathrm{i}\partial\!\!\!/\phi\gamma_*), \quad \delta F = \mathrm{i}\bar{\varepsilon}\partial\!\!\!/\psi. \tag{5.4}$$

The superderivatives are

$$D = \frac{\partial}{\partial\bar{\theta}} + \mathrm{i}\partial\!\!\!/\theta, \quad \bar{D} = -\frac{\partial}{\partial\theta} - \mathrm{i}\bar{\theta}\partial\!\!\!/, \quad \{D, \bar{D}\} = -2\mathrm{i}\partial\!\!\!/ \tag{5.5}$$

and the action is given by

$$\begin{aligned} S &= \int \mathrm{d}^2x\, \mathrm{d}\theta\, \mathrm{d}\bar{\theta} \left(\frac{1}{2}\bar{D}\Phi\gamma_* D\Phi + W(\Phi)\right) \\ &= \int \mathrm{d}^2x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\mathrm{i}}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}F^2 + \frac{1}{2}W''(\phi)\bar{\psi}\gamma_*\psi - W'(\phi)F\right]. \end{aligned} \tag{5.6}$$

To show the invariance under the above SuSy transformations we need Fierz identities derived from

$$\psi\bar{\chi} = -\frac{1}{2}\bar{\chi}\psi - \frac{1}{2}\gamma_\mu(\bar{\chi}\gamma_\mu\psi) - \frac{1}{2}\gamma_*(\bar{\chi}\gamma_*\psi) \tag{5.7}$$

as well as the symmetry relations (cf. appendix A)

$$\bar{\psi}\chi = -\bar{\chi}\psi, \ \bar{\psi}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\psi \quad \text{and} \quad \bar{\psi}\gamma_*\chi = \bar{\chi}\gamma_*\psi. \tag{5.8}$$

The Euclidean off-shell action is unstable. However, after the auxiliary fields have been integrated out, the on-shell action is stable.

5.2 The supersymmetric flow equations

For the derivation of the flow equation we proceed along the lines of supersymmetric quantum mechanics in the last chapter. Therefore only the differences are discussed here. For the products of covariant derivatives the following relations,

$$\left(\frac{1}{2}\bar{D}\gamma_* D\right)^{2n} = \frac{\mathrm{i}}{2}\bar{D}\partial\!\!\!/ D\left(\partial^2\right)^{n-1} \quad \text{and} \quad \left(\frac{1}{2}\bar{D}\gamma_* D\right)^{2n+1} = \frac{1}{2}\bar{D}\gamma_* D\left(\partial^2\right)^{n}, \tag{5.9}$$

hold with ∂^2 being the Laplacian. These relations are the key feature in the construction of the supersymmetric cutoff action. The most general function quadratic in the superfields that contains only covariant derivatives is the superspace integral of

$$\frac{1}{2}\Phi\bar{D}\left(\partial\!\!\!/\tilde{r}_1\left(k,-\partial^2\right) - \gamma_* r_2(k,-\partial^2)\right)D\Phi \tag{5.10}$$

with $r_1\left(k,-\partial^2\right) = q^2\tilde{r}_1\left(k,-\partial^2\right)$. Written out in components and in momentum space the cutoff action takes the form

$$\Delta S_k = \frac{1}{2}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\left[r_2\left(k,q^2\right)q^2\phi^2 + r_2\left(k,q^2\right)F^2 - 2r_1\left(k,q^2\right)F\phi + \bar{\psi}\left(q\!\!\!/ r_2\left(k,q^2\right) + \gamma_* r_1\left(k,q^2\right)\right)\psi\right]. \tag{5.11}$$

In matrix notation the regulator reads

$$R_k^{\mathrm{B}} = \begin{pmatrix} q^2 r_2\left(k,q^2\right) & -r_1\left(k,q^2\right) \\ -r_1\left(k,q^2\right) & -r_2\left(k,q^2\right)\end{pmatrix} \quad \text{and} \quad R_k^{\mathrm{F}} = q\!\!\!/\, r_2\left(k,q^2\right) + \gamma_* r_1\left(k,q^2\right). \tag{5.12}$$

As in the SuSy-QM, $r_1\left(k,q^2\right)$ is a momentum-dependent mass term and $r_2\left(k,q^2\right)$ is a momentum-dependent deformation of the kinetic term.

5.3 The local potential approximation

To solve the flow equation we first employ the local potential approximation. As an ansatz for the effective action we use

$$\Gamma_k[\phi,F,\bar{\psi},\psi] = \int \mathrm{d}^2x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}F^2 + \frac{1}{2}W_k''(\phi)\bar{\psi}\gamma_*\psi - W_k'(\phi)F\right), \tag{5.13}$$

which is the classical action but with a scale-dependent superpotential. In the approximation of constant fields it is possible to calculate the scale dependent propagator $\left(\Gamma_k^{(2)} + R_k\right)^{-1}$. The inverse propagator reads

$$\Gamma_k^{(2)} + R_k = \begin{pmatrix} A & W_k''' e_1 \otimes \bar{\psi}\gamma_* \\ W_k''' \gamma_* \bar{\psi} \otimes e_1^T & B \end{pmatrix} \quad \text{with} \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.14}$$

where the operators on the diagonal are given by

$$A = \begin{pmatrix} q^2(1+r_2) - FW_k''' + \frac{1}{2}W_k^{(4)}\bar{\psi}\gamma_*\psi & -W_k'' - r_1 \\ -W_k'' - r_1 & -1 - r_2 \end{pmatrix}, \quad B = \mathrm{i}(1+r_2)\not{q} + \gamma_*(r_1 + W_k''). \tag{5.15}$$

Using the above relations, the inverse can be calculated, see [124] for the details. Inserting 5.15 into the flow equation yields for the first derivative of the superpotential

$$\partial_k W_k' = -W_k''' \int \frac{\mathrm{d}^2 q}{4\pi^2} \left(\frac{(1+r_2)(W_k''+r_1)}{(q^2(1+r_2)^2 + (W_k''+r_1)^2)^2}\partial_k r_1 + \frac{q^2(1+r_2)^2 - (W_k''+r_1)^2}{2(q^2(1+r_2)^2 + (W_k''+r_1)^2)^2}\partial_k r_2 \right), \tag{5.16}$$

where we have projected on the terms linear in F. Integration with respect to ϕ and dropping an irrelevant constant leads to the flow equation of the superpotential:

$$\partial_k W_k = \frac{1}{2}\int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{(r_2+1)\partial_k r_1 - (r_1 + W_k''(\phi))\partial_k r_2}{q^2(1+r_2)^2 + (W_k''+r_1)^2} \tag{5.17}$$

Again, a projection onto $\bar{\psi}\gamma_*\psi$ results in the same flow equation for the superpotential.

In contrast to supersymmetric quantum mechanics, potentials with dynamical supersymmetry breaking are of interest in this chapter, i. e. superpotentials or order $O(\phi^{2n+1})$. Therefore the mass-like regulator $r_1\left(k, q^2\right)$ does not screen potential zero modes of $W''(\phi)$, whose highest power is odd as well, but merely shifts them. We will set it to zero in the following.

For the local potential approximation we will use the simple regulator

$$r_1 = 0, \quad r_2 = \left(\frac{k}{|q|} - 1\right)\theta(k^2 - p^2). \tag{5.18}$$

Keep in mind, however, that in two dimensions this regulator will not be sufficient for the next-to-leading approximation, so that we will need a regulator which diverges stronger in the infrared in the second part of this chapter.

With the choice of the regulator introduced above it is possible to perform the momentum integrals analytically. This yields the flow equation

$$\partial_k W_k(\phi) = -\frac{k}{4\pi}\frac{W_k''(\phi)}{k^2 + W_k''(\phi)^2} \quad \Rightarrow \quad \partial_k W_k'(\phi) = -W_k'''(\phi)\frac{k}{4\pi}\frac{k^2 - W_k''(\phi)^2}{(k^2 + W_k''(\phi)^2)^2}. \tag{5.19}$$

Note that due to the construction, the flow equation ensures that the *superpotential* is convex in the infrared. However, this is not necessarily true for its derivative or the bosonic potential as can be seen from the equations (5.19). As a consequence the effective potential $V = \frac{1}{2}W'^2_k(\phi)$ calculated from the superpotential is not necessarily convex, especially in the regime with unbroken supersymmetry. This can be traced back to the truncation, where a potential for the auxiliary field and supersymmetric partner terms are neglected. In order to obtain the 'true' convex effective potential, such terms have to be taken into account.

Finally we would like to add that the flow equation for $W'_k(\phi)$ changes its sign for $W''^2_k(\phi) = k^2$. This sign change will give constraints on the fixed point solution discussed below.

5.4 Fixed-point analysis

Before we solve the above flow equation for a given bare superpotential at the ultraviolet cutoff Λ we investigate the fixed-point structure. It will turn out, however, that the picture obtained from the LPA will change in next-to-leading order approximations. In two dimensions only part of the fixed points are accessible in the LPA, that is at $\eta = -\partial_t \ln\left(Z_k^2\right) = 0$. This is also known from bosonic theories in two dimensions [126, 127]. In this respect the supersymmetric model will behave very similar to the bosonic theories .

Since a fixed-point study requires a scaling form of the flow equation it is rewritten in terms of dimensionless quantities $w_t(\varphi) = {}^{W_k(\phi)}/_{k}$ and $t = \ln\left({}^{k}/_{\Lambda}\right)$. The two-dimensional field ϕ is dimensionless. To keep the notation consistent with the following chapters we use $\varphi = \phi$ in the dimensionless flow equations. The flow of the dimensionless potential reads

$$\partial_t w_t(\varphi) + w_t(\varphi) = -\frac{1}{4\pi}\frac{w_t''(\varphi)}{1 + w_t''(\varphi)^2}. \tag{5.20}$$

Fixed points are characterised by the condition $\partial_t w_*(\varphi) = 0$. The dimensionless equation for the first derivative is

$$\partial_t w_t'(\varphi) + w_t'(\varphi) = -\frac{1}{4\pi}\frac{w_t'''(\varphi)}{1 + w_t''(\varphi)^2} + \frac{2}{4\pi}\frac{w_t''(\varphi)w_t'''(\varphi)^2}{(1 + w_t''(\varphi)^2)^2}. \tag{5.21}$$

We first solve the equation with a polynomial approximation before the solution to the full nonlinear differential equation is considered.

5.4.1 Polynomial approximation

The flow equation for w' can always be expanded around its minimum $\varphi = 0$ even if the bosonic potential is a double-well potential. The polynomial approximation is justified for small values of the field φ.

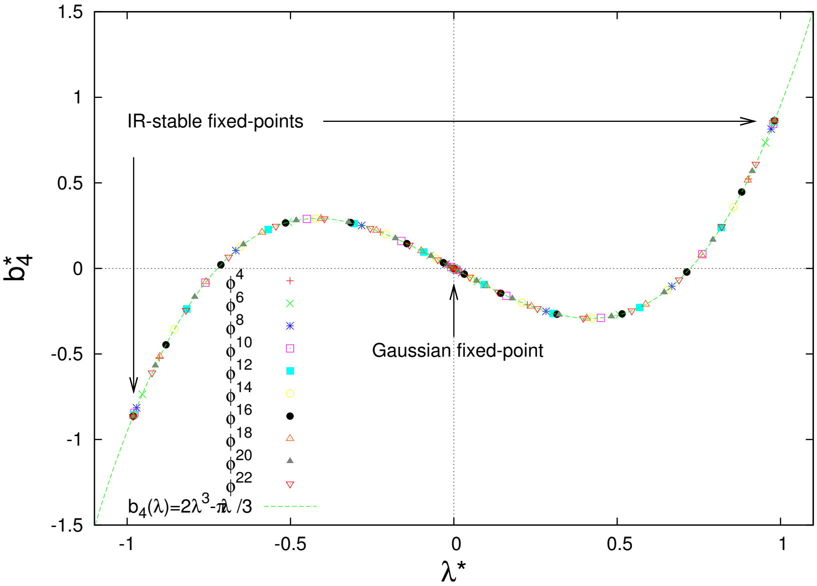


Figure 5.1: Projection of the coefficients of all fixed points for different truncations on the plane of the couplings λ and b_4.

At the ultraviolet cutoff we choose $W_\Lambda = \bar{\lambda}_\Lambda(\phi^2 - \bar{a}_\Lambda^2)$ and we employ the expansion $w_t'(\varphi) = \lambda_t(\varphi^2 - a_t^2) + \sum_{i=2}^{N} b_{2i,t}\varphi^{2i}$. An even potential remains even under the flow and thus only even terms have to be considered in the ansatz. The dimensionless couplings $\lambda_{i,t}, b_{i,t}$ are related to the bare, dimensionful ones via $\lambda_t = \bar{\lambda}/k$ and $b_{i,t} = \bar{b}_i/k$. $\bar{a}$ is dimensionless, therefore $a_t = \bar{a}_k$ holds. The flow equations in this approximation are

$$
\begin{aligned}
\partial_t a_t^2 &= \frac{1}{2\pi} - \frac{6\lambda_t^2 \cdot a_t^2}{\pi} + a_t^2 \frac{3b_{4,t}}{\pi\lambda_t}, \\
\partial_t \lambda_t &= -\frac{3b_{4,t}}{\pi} + \frac{6\lambda_t^3}{\pi} - \lambda_t, \\
\partial_t b_{4,t} &= -\frac{15 b_{6,t}}{2\pi} + \frac{60 b_{4,t}\cdot\lambda_t^2}{\pi} - \frac{40\lambda_t^5}{\pi} - b_{4,t}, \\
&\dots \\
\partial_t b_{2n,t} &= -\frac{(n+1)(n+2)}{4\pi} b_{2n+2,t} + f_{2n}\left(\lambda_t, b_{4,t}, \dots, b_{2n,t}\right).
\end{aligned}
\tag{5.22}
$$

The fixed-point equations are obtained by setting the left hand side equal to zero. The system of coupled equations has a triangular form and can be solved iteratively. From the equation for $\partial_t\lambda$ we can read off that all fixed-point couplings have to obey a relation between λ and b_4. This is shown in figure 5.1. The system of N equations yields $2N + 1$ real fixed points. One is the Gaußian fixed point with all couplings equal to zero, the other ones come in pairs due to the underlying $\mathbb{Z}_2$ symmetry. The largest root of the system of equations turns out to be the infrared stable fixed point. With increasing order of truncation it converges to $|\lambda_{\text{crit}}| = 0.982$. For the convergence behaviour see table 5.1 where the coefficients of the fixed point are shown for different truncations. All other roots are bounded by $|\lambda^*| = |\lambda_{\text{crit}}|$.

	Coefficients at infrared fixed point							
$2n$	λ^*	b_4^*	b_6^*	b_8^*	b_{10}^*	b_{12}^*	b_{14}^*	b_{16}^*
2	0.7236							
4	0.9019	0.5227						
6	0.9535	0.7354	0.8372					
8	0.9711	0.8148	1.199	1.694				
10	0.9777	0.8451	1.345	2.420	3.801			
12	0.9802	0.8570	1.402	2.716	5.401	9.030		
14	0.9812	0.8617	1.425	2.836	6.054	12.77	22.23	
16	0.9816	0.8636	1.435	2.884	6.318	14.29	31.33	56.11

Table 5.1: The coefficients of the infrared-stable fixed-point potential for different truncation orders.

Stability analysis and critical exponents

The coupling constants at the fixed point and the radius of convergence are regulator dependent, hence they are not physical quantities. The critical exponents are, however, universal and classify the fixed points. They are defined as the negative eigenvalues θ^I of the stability matrix at the fixed point (see section 2.2).

The flow equation for a_t^2 reads

$$\partial_t a_t^2 = \frac{1}{2\pi} - a_t^2 - \left(\frac{a_t^2}{\lambda_t}\right) \partial_t \lambda_t. \tag{5.23}$$

At any fixed point the $(0,0)$-component of the stability matrix is ${B_0}^0 = -1$. Due to the triangular form of the system (5.22) it follows that ${B_{i>1}}^0 = 0$. Therefore a_t^2 is always an eigendirection of ${B_i}^j$ with eigenvalue $\theta^0 = 1$ independent of the regulator. The value of this critical exponent receives corrections at higher orders in the derivative expansion, but it will always remain to be an eigendirection because couplings of higher order do not contribute to the flow equation of a_t^2. This implies that the superpotential in the LPA always has at least one infrared-unstable direction. All other $2N-1$ fixed points turn out to have more than one infrared-unstable direction.

The critical exponent $\nu_W^{-1} = (\theta^0) = 1$ *does not* correspond to the scaling exponent in the correlation length, unlike in the bosonic Ising model. Rather, the critical exponent governs how the bosonic mass scales with the RG scale. It also plays a role in describing the phase diagram.

For a polynomial approximation to order $2n = 16$ the critical exponents are calculated with two different regulators

$$r_2 = \left(\frac{k}{|q|} - 1\right) \theta \left(\frac{k^2}{q^2} - 1\right) \quad \text{and} \quad r_2 = \left(\frac{k^2}{q^2} - 1\right) \theta \left(\frac{k^2}{q^2} - 1\right). \tag{5.24}$$

Regulator	λ_*	Critical exponents θ^I								
$\left(k/p - 1\right)\theta(k^2/p^2 - 1)$	±0.9816	1	-1.54	-7.43	-18.3	-37.3	-68.9	-120	-204	-351
	±0.8813	1	6.16	-1.64	-9.82	-25.6	-52.5	-96.9	-170	-300
	±0.7131	1	21.4	4.37	-1.57	-11.1	-30.1	-63.3	-120	-223
	±0.5152	1	28.7	13.3	3.33	-1.39	-11.6	-32.8	-71.7	-145
	±0.3158	1	20.0+4.55i	20.0-4.55i	8.40	2.57	-1.14	-11.6	-34.3	-80.4
	±0.1437	1	11.2+9.02i	11.2-9.02i	8.63	5.19	1.95	-0.84	-11.1	-35.7
	±0.0322	1	4.20+1.18i	4.20-1.18i	2.86	2.72+6.47i	2.72-6.47i	1.47	-0.54	-10.5
	±0.0003	1	1.57+0.13i	1.57-0.13i	1.43+0.70i	1.43+0.70i	1.14	0.54+0.98i	0.54+0.98i	-0.22
	0	1	1	1	1	1	1	1	1	1
$\left(k^2/p^2 - 1\right)\theta(k^2/p^2 - 1)$	±1.6315	1	-1.31	-7.10	-19.3	-42.7	-84.8	-158	-285	-522
	±1.4399	1	5.43	-1.49	-10.1	-28.2	-61.8	-122	-227	-426
	±1.1463	1	19.5	4.07	-1.51	-11.8	-33.9	-75.7	-152	-298
	±0.8175	1	28.1	12.4	3.23	-1.39	-12.5	-37.3	-85.7	-182
	±0.4958	1	20.4+3.09i	20.4-3.09i	8.06	2.54	-1.16	-12.5	-38.9	-95.0
	±0.2232	1	11.9+8.85i	11.9-8.85i	8.69	5.07	1.96	-0.86	-12.0	-39.8
	±0.0490	1	4.27+1.14i	4.27-1.14i	2.91+6.63i	2.91-6.63i	2.84	1.47	-0.54	-11.1
	±0.0004	1	1.57+0.125i	1.57-0.125i	1.43+0.70i	1.43-0.70i	1.14	0.542+0.98i	0.542-0.98i	-0.22
	0	1	1	1	1	1	1	1	1	1

Table 5.2: Critical exponents θ^I (negative eigenvalues of the stability matrix) for a polynomial truncation at $2n = 16$ for the nine different fixed points in the local-potential approximation. For all positive critical exponents, a remarkable degree of universality is found, as these exponents for the different regulators differ by at most 10%. The subleading negative critical exponents show larger variations.

The results are shown in table 5.2. Positive exponents, which belong to infrared-unstable directions, are highlighted in gray. The variation in the positive critical exponents is of order 10% or less. This confirms the expected regulator independence. The fixed points can be labelled by the slope λ_* of the dimensionless superpotential $w'(\phi^2)$. The largest slope corresponds to the fixed point with the most infrared-stable directions. As the slope decreases, the number of infrared-unstable directions increases. The Gaußian fixed point with $\lambda_* = 0$ has no infrared-stable directions. Thus we conclude that each fixed point defines a different non-perturbative renormalised Wess-Zumino model in the ultraviolet in two dimensions. If these fixed points survive to higher orders, the number of physical parameters increases for these fixed points.

At a fixed point the relevant directions are infrared repulsive and the fine tuning of the relevant direction to the fixed point brings the system to its critical point. For the maximally infrared-stable fixed point the unstable direction a_t^2 is the only parameter that has to be fine tuned. In this respect a_t^2 is similar to the temperature in Ising like systems or to the mass in $O(N)$ models. In the domain of the maximally infrared-stable fixed point the tuning of a_t^2 distinguishes between the supersymmetric broken and unbroken phase. In the domain of N relevant directions there is an N-dimensional hypersurface that separates the supersymmetric and non-supersymmetric phase. Unlike the Ising-like systems a_t^2 does not influence the higher couplings because it can be expanded around $\varphi = 0$. For this reason the remaining couplings are attracted towards the maximally infrared-stable fixed point. As long as the polynomial approximation is valid the flow of w'' is governed by the maximally infrared-stable fixed point. This is also the case in higher dimensions (cf. chapter 6).

This behaviour remains unchanged if higher orders in the supercovariant derivative expansion are taken into account because the superpotential (or its derivative) does not couple to higher orders of the auxiliary field.

5.4.2 Solving the nonlinear differential equation

To go beyond the approximation of small fields we have to consider the nonlinear differential equation. The case of two dimensions is special because the field is dimensionless and the term $(d-2)\varphi w''(\varphi)$ is not present in the left hand side of the fixed point equation

$$\partial_t w_t'(\varphi) + w_t'(\varphi) = -\frac{1}{4\pi}\frac{w_t'''(\varphi)}{1+w_t''(\varphi)^2} + \frac{2}{4\pi}\frac{w_t''(\varphi)w_t'''(\varphi)^2}{(1+w_t''(\varphi)^2)^2}. \tag{5.25}$$

The right hand side contains w'' as highest derivative. Due to this, an infrared-stable solution is found if the fixed point equation for w'' is considered. For simplicity of notation $u = w''(\varphi)$ is introduced. The fixed point equation then reads

$$(1-u^2)(1+u^2)u'' = 2u'^2\,(3-u^2)\,u - (1+u^2)^3\,4\pi u. \tag{5.26}$$

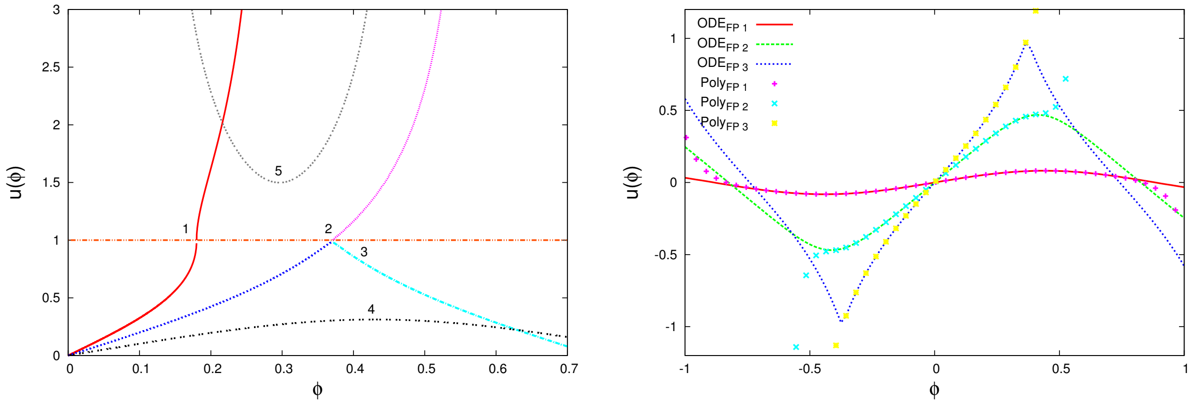


Figure 5.2: *Left panel:* All types of possible solutions to the fixed-point differential equation in local-potential approximation: (1) $\gamma > \gamma_{\text{crit}}$, (2) $\gamma = \gamma_{\text{crit}}$, finite φ range, (3) $\gamma = \gamma_{\text{crit}}$, oscillating solution, (4) $\gamma < \gamma_{\text{crit}}$, oscillating solution, (5) solution with just one extremum . *Right panel:* Comparison between the numerical solution to the differential equation (ODE) and the polynomial approximation (Poly) to 16th order for three different fixed points. The fixed points FP1, FP2, and FP3 have the initial slope $\gamma = 0.287, 1.4262$ and 1.963. The fixed point FP3 is the maximally infrared-stable fixed point.

In [124] it is proven that the nonlinear equation has oscillating solutions with $|u| \leq 1$ if the starting slope $u'(0) = \gamma \leq \gamma_{\text{crit}}$. For $\gamma > \gamma_{\text{crit}}$ there are diverging solutions that are confined to a finite range of the field φ. In figure 5.2, left panel, we depict all possible types of solutions. The oscillating solution is shown in figure 5.2, right panel, for three different slopes together with the Taylor expansion from the polynomial approximation. The Taylor expansion is a good approximation for the first half of one period.

Now we can make the connection between the fixed points found in the polynomial approximation and from the solution of the differential equation. In case of $\gamma \leq \gamma_{\text{crit}}$ the solutions to the polynomial approximation belong to the Taylor expansion of the oscillating solutions. The solutions are bounded by $\gamma_{\text{crit}} = 2\lambda_{\text{crit}}$ corresponding to the infrared-stable fixed point. Therefore there are infinitely many sine-Gordon type solutions to the fixed point equation.

Due to the factor $(1 - u^2)$ in equation (5.26) the left hand side of equation (5.26) vanishes at $u = 1$. For a regular solution this implies that the right hand side has to vanish as well. From this we find a condition for the slope at the critical point, namely that it has to be equal to $u'(\varphi_{\text{crit}}) = \pm\sqrt{8\pi}$. At φ_{crit} the solution can either continue with the same slope or it can be reflected which leads to an oscillating solution. This is different from the situation in three dimensions, where a similar condition arises. However, there it determines a unique solution without the possibility of reflection (cf. chapter 6).

5.4.3 Fixed points at next-to-leading order

It is known from the fixed-point analysis of two-dimensional bosonic theories [126, 127] that in the LPA with $\eta = 0$ only oscillating solutions and solutions that are defined over a finite φ range are accessible. For $\eta \neq 0$ a term $\sim \eta\varphi w''$ enters on the left hand side of the flow equation, changing the picture when the next-to-leading order is taken into account. Now, regular non-periodic solutions with a polynomial asymptotic behaviour emerge. In this case the approximation for the effective action is

$$\Gamma_k[\phi, F, \bar{\psi}, \psi] = \int d^2x \left[Z_k^2 \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}F^2 \right) + \frac{1}{2}W_k''(\phi)\bar{\psi}\gamma_*\psi - W_k'(\phi)F \right]. \quad (5.27)$$

As cutoff action we choose

$$\Delta S_k = \frac{1}{2}\int \frac{d^2q}{(2\pi)^2} Z_k^2 \left[r_2 q^2\phi^2 + r_2F^2 - 2r_1F\phi + \bar{\psi}(q\!\!\!/ r_2 + \gamma_* r_1)\psi \right]. \quad (5.28)$$

With this, the flow equation for the superpotential is obtained by a projection onto the terms linear in the auxiliary field and by an integration with respect to ϕ.

$$\partial_k W_k = \frac{1}{2}\int \frac{d^2q}{4\pi^2} \left[\frac{(1+r_2)Z_k^2\,\partial_k(r_1Z_k^2)}{Z_k^4 q^2(1+r_2)^2 + (W_k'' + r_1Z_k^2)^2} - \frac{(W_k'' + r_1Z_k^2)\partial_k(r_2Z_k^2)}{Z_k^4 q^2(1+r_2)^2 + (W_k'' + r_1Z_k^2)^2} \right]. \quad (5.29)$$

The flow equation of the wave-function renormalisation is obtained by a projection onto the terms quadratic in the auxiliary field. As only a field-independent wave-function renormalisation is considered, the flow equation for the wave-function renormalisation can additionally be projected onto $\phi = 0$:

$$\partial_k Z_k^2 = -W_k'''(\phi)^2 Z_k^2 \int \frac{d^2p}{4\pi^2}(1+r_2)\frac{1}{\left(Z_k^4 p^2(1+r_2)^2 + (W_k'' + r_1Z_k^2)^2\right)^3} \times \quad (5.30)$$
$$\left[2Z_k^2\left(W_k''(\phi) + r_1Z_k^2\right)(1+r_2)\,\partial_k(r_1Z_k^2) + \left(Z_k^4p^2(1+r_2)^2 - \left(W_k''(\phi) + r_1Z_k^2\right)^2\right)\partial_k(r_2Z_k^2) \right]\Big|_{\phi=0}$$

Taking into account the running of a wave-function renormalisation the simple regulator we have used before leads to artificial singularities in the infrared. For this reason, we use a different regulator

$$r_1 = 0 \quad \text{and} \quad r_2 = \left(\frac{k^2}{p^2} - 1\right)\theta\left(k^2 - p^2\right) \quad (5.31)$$

in the subsequent calculations. After the rescalings $\chi = Z_k\phi$, $\mathcal{W}_k(\chi) = W_k(\phi)$ and $\mathcal{W}_k' = W_k'Z_k^{-1}$, $\mathcal{W}_k'' = W_k''Z_k^{-2}, \ldots$ the flow equations read:

$$k\partial_k\mathcal{W}_k(\chi) - \frac{\eta}{2}\chi\mathcal{W}_k' = -\frac{\eta\,k^2}{8\pi\mathcal{W}_k''} + \frac{(\eta-2)k^2\mathcal{W}_k''^2 + \eta k^4}{8\pi\mathcal{W}_k''^3}\ln\left(1 + \frac{\mathcal{W}_k''^2}{k^2}\right), \quad (5.32)$$
$$\eta = \frac{k^2}{4\pi}\left(\frac{\mathcal{W}_k'''}{\mathcal{W}_k''^2}\right)^2 \left[\frac{\eta\mathcal{W}_k''^2}{\mathcal{W}_k''^2 + k^2} - \eta\ln\left(1 + \frac{\mathcal{W}_k''^2}{k^2}\right) + \frac{2\mathcal{W}_k''^4}{(\mathcal{W}_k''^2 + k^2)^2} \right]\Bigg|_{\chi=0}.$$

In terms of the dimensionless superpotential $\mathfrak{w}(\chi) = \mathcal{W}(\chi)/k$, this can be rewritten as

$$\partial_t \mathfrak{w}_k(\chi) - \frac{\eta}{2}\chi\mathfrak{w}'_k + \mathfrak{w}_k = -\frac{\eta}{8\pi\mathfrak{w}''_k} + \frac{(\eta-2)\mathfrak{w}''^2_k + \eta}{8\pi\mathfrak{w}''^3_k}\ln\left(1 + \mathfrak{w}''^2_k\right), \tag{5.33}$$

$$\eta = \frac{1}{4\pi}\left(\frac{\mathfrak{w}'''_k}{\mathfrak{w}''^2_k}\right)^2 \left[\frac{\eta\mathfrak{w}''^2_k}{\mathfrak{w}''^2_k + 1} - \eta\ln\left(1 + \mathfrak{w}''^2_k\right) + \frac{2\mathfrak{w}''^4_k}{(\mathfrak{w}''^2_k + 1)^2}\right]\Bigg|_{\chi=0}. \tag{5.34}$$

Polynomial approximation

For a polynomial solution we use the expansion

$$\mathfrak{w}'_t(\chi) = \lambda_t(\chi^2 - a_t^2) + \sum_{n=1}^{N} b_{2n,t}\chi^{2n}. \tag{5.35}$$

The limit $\chi \to 0$ on the right-hand side of equation (5.34) exists and the equation can be resolved with respect to η. This yields

$$\eta = \frac{4\lambda^2}{\lambda^2 + 2\pi}. \tag{5.36}$$

A polynomial approximation of equation (5.33) yields the flow equation for the coupling a_t^2:

$$\begin{aligned}\partial_t a_t^2 &= \frac{1}{8\pi}(4-\eta_t) - \eta_t a_t^2 - \frac{a_t^2}{\lambda_t}\left(\frac{3\left(\lambda_t^3 - b_{4,t}\right)}{\pi} - \frac{\left(2\lambda_t^3 - 3b_{4,t}\right)\eta_t}{4\pi}\right)\\ &= \frac{1}{2\pi}\left(1 - \frac{\eta}{4}\right) - \left(1 - \frac{\eta}{2}\right)a_t^2 - \frac{a_t^2}{\lambda_t}\,\partial_t\lambda_t\end{aligned} \tag{5.37}$$

As in the LPA a_t^2 is an eigendirection of the stability matrix and the corresponding critical exponent is given by

$$\theta^0 = \left(1 - \frac{\eta}{2}\right) \quad\Rightarrow\quad \nu_W \equiv \frac{1}{\theta^0} = \frac{2}{2-\eta}. \tag{5.38}$$

This relation is called *superscaling relation* because it relates the anomalous dimension with the critical exponent. In Ising-like systems the main thermodynamic exponents $(\alpha, \beta, \gamma, \delta)$ are related among each other by scaling relations. They can be deduced from the exponents ν and η of correlation functionsby hyperscaling relations, but no other connection between these exponents exists. In this respect the superscaling relation is specific to supersymmetric theories as it provides a connection between ν_W and η that does not exist in non-supersymmetric theories.

The superscaling relation is exact at next-to-leading order, and although it might receive corrections from higher-order derivative operators beyond next-to-leading order it still constitutes a new relation between η and ν. In fact, quantitative corrections beyond next-to-leading order are expected to be quite large because the anomalous dimension is large. The value for the critical

$2n$	2	4	6	8	10	12	14
η	0.3284	0.4194	0.4358	0.4386	0.4388	0.4387	0.4386
$2-\eta/2$	0.8358	0.7903	0.7821	0.7807	0.7806	0.78065	0.7807
$1/\nu_W$	0.8358	0.7903	0.7821	0.7807	0.7806	0.78065	0.7807

Table 5.3: Numerical verification of the superscaling relation (5.38): anomalous dimension η and the critical exponent ${\nu_W}^{-1}$ of a^2 for increasing orders in a polynomial truncation evaluated for the maximally infrared-stable fixed point. They converge fast with increasing order of the truncation.

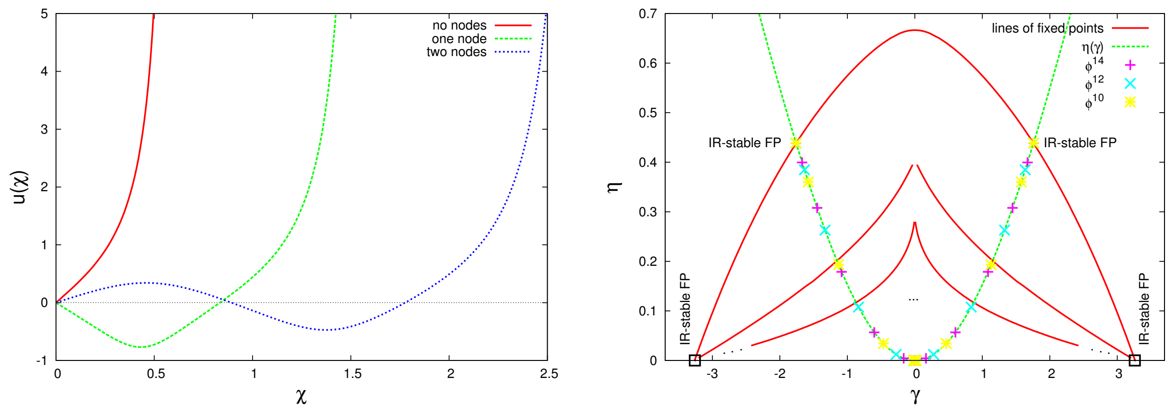


Figure 5.3: *Left panel:* Regular potentials for $\eta = 0.1$. The asymptotic behaviour of these potentials is given by $\mathfrak{w}''(\chi) = \chi^{18}$. *Right panel:* Lines of fixed points in the η-γ plane (solid curves) and the anomalous dimension as a function of $\gamma = 2\lambda$ obtained from equation (5.36) (dotted curve). Also displayed are the fixed point solutions obtained from a polynomial approximation of equation (5.33) and (5.34) for different truncations.

exponent shows a sufficiently fast convergence with the order of the polynomial approximation, as can be seen from the first row in table 5.3. Additionally we give in this table a numerical verification of the superscaling relation.

The superscaling relation and its consequences for the infrared flow of the masses in the regime with broken supersymmetry will be discussed later.

Fixed points from the nonlinear flow equation

In the limit of large χ the right hand side of the fixed point equation following from equation (5.33) is subdominant, implying the asymptotic behaviour

$$\mathfrak{w}_* \sim \chi^{2/\eta} \tag{5.39}$$

for the superpotential. Such solutions are shown in figure 5.3 (left panel) for $\eta = 0.1$. For $\eta = 0$ the asymptotic potential grows faster than any polynomial. This agrees with the results in the previous section. For $\eta \neq 0$ a new class of solutions emerge that are regular over the whole φ-range.

The fixed point equation for the second derivative of the superpotential reads[1]

$$\frac{u''}{4\pi}\left[\left(\frac{\eta(3+u^2)}{2u^4}-\frac{1}{u^2}\right)\ln\left(1+u^2\right)+\frac{2}{1+u^2}-\frac{3\eta}{2u^2}\right]$$
$$=(\eta-1)u+\frac{\eta\chi}{2}u'+\frac{u'^2}{2\pi}\left[\frac{1+3u^2}{u(1+u^2)^2}-\frac{\eta(3+2u^2)}{u^3(1+u^2)}+\left(\frac{\eta(6+u^2)}{2u^5}-\frac{1}{u^3}\right)\ln\left(1+u^2\right)\right]. \quad (5.40)$$

As in the case of the LPA the vanishing of the factor multiplying u'' at some χ_{crit} yields a condition on the slope u' at χ_{crit}. Due to the term $\frac{1}{2}\eta\chi u'$ the slope is no longer independent of χ_{crit}, however.

In order to investigate this equation we first consider η as a free parameter in equation (5.40). This is similar to the way Neves et. al. [127] investigated the two-dimensional bosonic models. As initial conditions $u(0) = 0$ we use and $u'(0) = \gamma = 2\lambda$ because we are only interested in odd solutions for u. Integrating the equation with a generic slope for a given η ends in a singularity because the factor multiplying u'' becomes zero at some point. Nevertheless it is possible to find regular solutions by fine tuning the slope at the origin. For $\eta = 0.1$ three solutions are shown in the left panel of figure 5.3.

All regular solutions define lines of fixed points in the η–γ plain. These lines are shown in the right panel of figure 5.3. The largest value of η for which a regular solution exists is $\eta = 2/3$. For this η the potential behaves as $u \sim \chi$. For $0 < \eta < 2/3$ we can read off from the monotony of the functions that the factor multiplying u'' in equation (5.40) has only one root at some $\chi = \chi_{\text{crit}}$. By fine tuning of the starting slope we can achieve that the right hand side vanishes.

The outermost curve in figure 5.3 corresponds to solutions $u(\chi)$ with no nodes, the next curve to solutions with one node and the third curve to solutions with two nodes. Solutions with more nodes can be found for small η and γ. We also display η as a function of γ,

$$\eta(\gamma) = \frac{4\gamma^2}{\gamma^2 + 8\pi}, \quad (5.41)$$

in figure 5.3, right panel. Its intersections with the lines of fixed points pick out the solutions that satisfy the fixed-point equation (5.40) and the equation for the anomalous dimension. This can be observed in the polynomial approximation of the fixed-point equation which converges to the maximally infrared-stable solution with $\eta = 0.4386$ and $\gamma = \pm 1.759$.

The point $\eta = 0$ and $\gamma = \pm 3.529$ where all curves meet corresponds to solutions that diverge for a finite value of φ. These solutions were discussed in the local potential approximation. As the slope at $u = 1$ does not depend on φ_{crit} for $\eta = 0$, the solution either diverges without any cusps such that it lies on the outermost curve or it oscillates a number of times and then diverges such that it lies on one of the inner curves with one node or more.

The solutions at leading and at next-to-leading order seem to be qualitatively very different.

[1] Again, we use the abbreviation $u = \mathfrak{w}''_*$.

The leading order solutions are either oscillatory, the infrared-stable solution even has cusps, or they have a compact target space whereas the next-to-leading order solutions allow for solutions that behave like $\mathfrak{w} \sim \chi^{2/\eta}$ for large fields but exhibit oscillatory behaviour for small fields.

An important parameter to understand the connection between leading and next-to-leading order is the anomalous dimension η from which the asymptotic polynomial behaviour originates. It has been derived from a small field expansion around $\chi = 0$, but a field-dependent wave-function renormalisation is expected to behave as $Z_k(\phi \to \infty) \to 1$ and correspondingly $\eta(\chi \to \infty) \to 0$. Thus it is reasonable to expect that the true asymptotic behaviour of the fixed-point potential is bound to lie between the one from leading and next-to-leading order, i. e. it will show a stronger divergence at $\chi \to \infty$ than the one predicted by the next-to-leading order results.

Before concluding this section on the fixed points a short discussion of the situation in two-dimensional bosonic theories is in order. T. Morris [126] as well as R. Neves et. al. [127] discovered that solutions which have a polynomial asymptotic behaviour can only be found for non-vanishing anomalous dimensions. Morris demonstrated that at next-to-leading order the fixed-point solutions, which are classified by their number of nodes, correspond to conformal field theories described by Zamolodchikov [128].

We expect the fixed-point solutions discovered here to correspond to conformal theories as well. For the two-dimensional $\mathcal{N} = 2$ Wess-Zumino model it has recently been demonstrated with lattice simulations that the infrared fixed point of this model describes $\mathcal{N} = 2$ superconformal minimal models [129].

Having studied the structure of the general fixed-point solutions now we will investigate a specific model.

5.5 The Gaußian Wess-Zumino model

In this section we are interested in a quadratic perturbation to the Gaußian fixed point at the ultraviolet cutoff Λ. We take it to be of the form $W'_\Lambda = \bar{\lambda}_\Lambda(\phi^2 - \bar{a}^2_\Lambda)$. This means that we consider an asymptotically free theory with infinitely many couplings set to zero at the ultraviolet cutoff scale. The ultraviolet cutoff is not removed in the following. Instead the non-universal bare quantities are used.

The regulator dependence of the RG trajectories and the large variation in the values of non-universal quantities for different regulators makes it difficult to compare the results to lattice calculations because the lattice regularisation of the same physical system might lead to a different lattice cutoff $\Lambda = {}^{\pi}/a_{\text{lat}}$. For this reason we make such a comparison only on a qualitative level.

From a different viewpoint the regulator choice can be interpreted as belonging to the theory itself, namely that the initial conditions, the perturbation at the cutoff scale and the regulator determine the RG trajectory at a finite scale Λ. The problem of this interpretation is that the

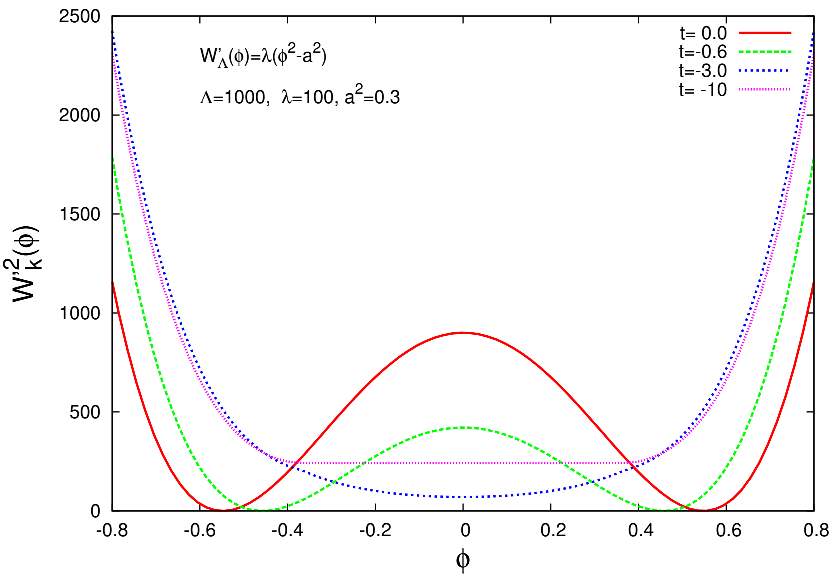


Figure 5.4: Flow of the superpotential with the starting conditions $W'_\Lambda(\phi) = \bar{\lambda}_\Lambda(\phi^2 - \bar{a}^2_\Lambda)$, $\Lambda = 1000$, $\bar{\lambda}_\Lambda = 100$, $\bar{a}^2_\Lambda = 0.3$.

couplings of infinitely many operators have to be adjusted when the cutoff scale is changed along the lines of constant physics. In order to find out how much these operators actually affect the flow of the superpotential, we vary the cutoff scale Λ. For a given Λ, the couplings in $W'_\Lambda = \bar{\lambda}_\Lambda(\phi^2 - \bar{a}^2_\Lambda)$ are adjusted such that fixed reference couplings a_{Λ_0} and λ_{Λ_0} are obtained at a reference scale Λ_0, ignoring higher-order couplings. For large enough Λ, the solutions of the flow equation show that the dependence on the ultraviolet cutoff scale of the ground state energy at $k = 0$ is small.

To calculate the flow of the superpotential we have to pay attention to the diverging derivatives in the infrared-stable solutions at ϕ_{crit} of the fixed point equations. For $\phi \in [-\phi_{\text{crit}}, \phi_{\text{crit}}]$ we use the polynomial approximation and outside of this regime the partial differential equation. At the point ϕ_{crit} both solutions have to be matched which is achieved by taking the polynomial approximation at ϕ_{crit} as a boundary condition. The details of the calculation are discussed in [124]. An example for the flow of a superpotential with dynamical supersymmetry breaking is shown in figure 5.4.

In this truncation the effective potential is given by $\frac{1}{2}W'^2_{k\to 0}$. As already discussed in the chapter on supersymmetric quantum mechanics this potential is not guaranteed to be convex. To obtain a convex potential higher orders in the super-covariant derivative expansion have to be taken into account.

In the Wess-Zumino models there is a connection between supersymmetry breaking and the restoration of $\mathbb{Z}_2$ symmetry. If supersymmetry is unbroken the bosonic potential is a double-well potential and the $\mathbb{Z}_2$ symmetry is broken. In the phase with broken supersymmetry the bosonic potential is a single-well potential and the $\mathbb{Z}_2$ symmetry is restored.

If the bare potential is chosen in the phase with unbroken supersymmetry the scalar potential

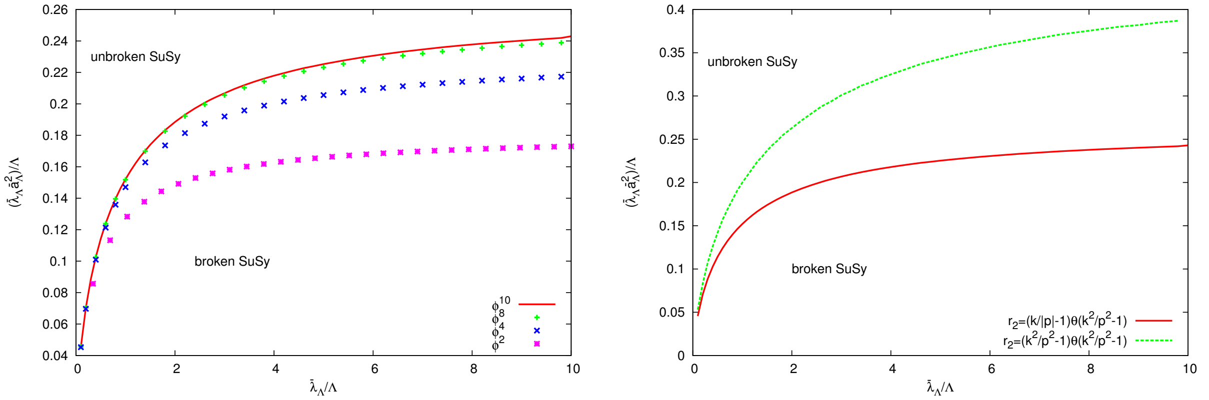


Figure 5.5: *Left panel:* Phase diagram in the space of the dimensionless couplings specified at the cutoff scale Λ for different truncations. *right panel:* Comparison between two different regulators

has two minima with $V = 0$. These minima approach $\phi = 0$ during the flow. At the phase transition there is just one minimum with $\phi = 0$ and $V = 0$ such that the phase transition for SuSy breaking and the restoration of the $\mathbb{Z}_2$ symmetry coincide. In the next chapter this relation is used in order to consider supersymmetry breaking at finite temperature.

5.5.1 Phase diagram

We will stay in the LPA to discuss the phase diagram. For a superpotential of order $W_k \sim O(\phi^{2n+1})$ supersymmetry is broken or unbroken depending on the parameter $\bar a_\Lambda^2$. As the criterion for supersymmetry breaking we take a non-vanishing ground state energy of the effective potential. This can only be the case if $W_k'(\phi)$ is nonzero on the whole ϕ-range. As the minimum of $W_k'(\phi)$ is located at $\phi = 0$ the polynomial approximation can be used to calculate the phase diagram. The minimum is given by $W_k'(0) = k\lambda_t a_t^2$. It freezes out because the coupling λ_t flows to its infrared fixed point λ_* as discussed in the previous section and $a^2 \sim \pm k^{-1}$ depending on the value at the cutoff scale. This is a direct consequence of the fact that a_t^2 at the fixed point is governed by the critical exponent $\theta^0 = 1$. The value of $\bar a_\Lambda$ at the cutoff scale determines whether a system that is broken at the cutoff scale remains in the broken regime ($a_t^2 \to +\infty$) or flows to the unbroken phase ($a_t^2 \to -\infty$). The change in the sign is taken as the signal for the phase transition. The phase transition line consists of those values $(\bar a_\Lambda^2, \bar\lambda_\Lambda)$ at which the sign change occurs. In figure 5.5, left panel, the phase diagram is shown. The values have been calculated with a truncation up to ϕ^{10}. The convergence is fast as the truncation order is increased.

In the strong coupling limit $\bar\lambda_\Lambda \to \infty$ there is a maximal value for $\bar\lambda_\Lambda \bar a_\Lambda^2$ above which supersymmetry can never be broken dynamically. From a numerical solution at a high-order truncation an estimate for this value is $\bar\lambda_\Lambda \bar a_\Lambda^2|_{\text{crit}}\Lambda^{-1} \simeq 0.263$. This is in agreement with qualitative results from the literature [116, 119].

We have additionally calculated the phase diagram with the regulator

$$r_2 = \left(\frac{k^2}{q^2} - 1\right) \theta \left(\frac{k^2}{q^2} - 1\right). \tag{5.42}$$

A comparison between the two regulators is shown in the right panel of figure 5.5. We observe a strong dependence of the values of the phase transition on the regulator. The numerical values differ by a factor of approximately two. Due to the regulator dependence a quantitative comparison between non-universal quantities in the FRG and in the lattice calculation [119], where just one point in the phase diagram was calculated, is not sensible. More lattice points are needed to compare dimensionless ratios that are less affected by scheme dependencies.

As the ground state energy and the fermionic mass are order parameters for the phase transition from broken to unbroken supersymmetry, they should exhibit a scaling behaviour near the phase transition. However, in the considered truncation such a scaling behaviour cannot be found. The auxiliary field is nonzero in the broken phase, therefore its expectation value yields a field valued order parameter. We expect fluctuations of the auxiliary field to play an important role near the critical point. The fluctuations might establish a scaling behaviour. To describe these fluctuations, a potential for the auxiliary field must be included. Such terms come from higher orders in the super-covariant derivative expansion and therefore the quantitative description of the critical regime is a hard challenge to tackle in the framework of the flow equations.

5.5.2 Scaling of the mass term

As in supersymmetric quantum mechanics the curvature at the minimum of the effective potential is defined as the bosonic mass in the infrared limit $k \to 0$. For renormalised fields $\chi = Z_k \phi$ the bosonic potential takes the form

$$V_k\left(\chi\right) = \frac{\left(W_k'\left(\chi \cdot Z_k^{-1}\right)\right)^2}{2Z_k^2}. \tag{5.43}$$

The bosonic mass reads

$$m_k^2 = V_k''\left(\chi_{\rm min}\right), \quad V_k'\left(\chi_{\rm min}\right) = 0. \tag{5.44}$$

In the broken phase $\chi_{\rm min} = 0$ holds. The scalar mass is given by

$$m_k^2 = Z_k^{-4} W_k'(0) W_k'''(0) = 2k^2 \lambda_t \left|a_t^2\right|. \tag{5.45}$$

For $k \to 0$ the system flows to its infrared-stable fixed point $\lambda_t \to \lambda_*$ and $a_t^2 \sim k^{-1/\nu_W}$, implying the scaling behaviour

$$m_k^2 \sim k^{1+\eta/2}. \tag{5.46}$$

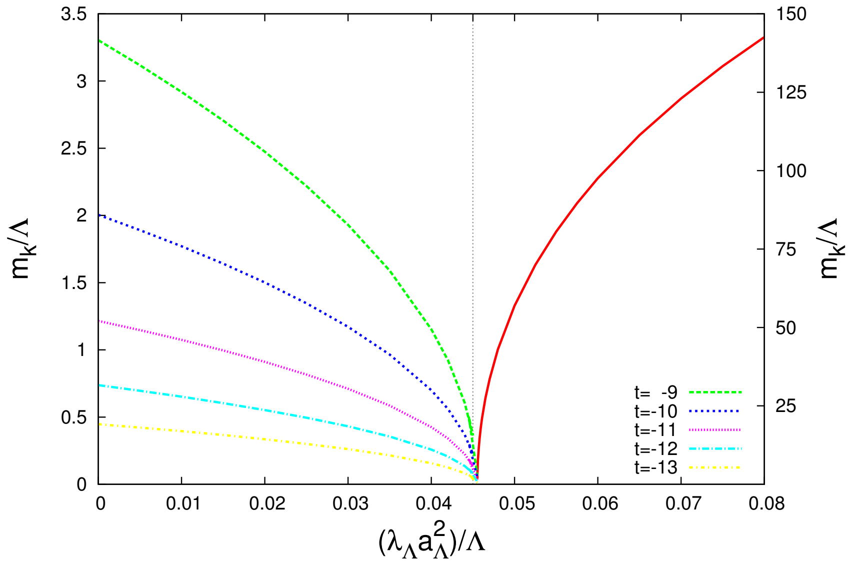


Figure 5.6: Renormalised mass at different scales k as a function of the initial condition $\bar{\lambda}_\Lambda \bar{a}_\Lambda^2$ at an initial coupling of $\bar{\lambda}_\Lambda = 0.1$. Note that the axes on the left and on the right hand side of the plot are scaled differently.

In the infrared-limit the scalar mass goes to zero for $\eta > -2$. Together with the Goldstino from supersymmetry breaking this leads to the conclusion that the Gaußian Wess-Zumino model is massless in both degrees of freedom.

Keep in mind, however, that the limit $k \to 0$ can never be realised in experiment but instead a cutoff scale is always involved. In lattice simulations this cutoff scale is the lattice volume. Hence we have shown that the bosonic mass is proportional to the cutoff scale involved in the measurement. First results from lattice simulations seem to confirm this conjecture [16].

The superscaling relation, together with $Z_k \sim k^{-\eta/2}$, causes the minimum of the superpotential to freeze out:

$$W'(0) = -\bar{\lambda}_k \bar{a}_k^2 = -k Z_k \lambda_t a_t^2 \sim k^{1-\eta/2} k^{-1/\nu_W} \to \text{const.} \tag{5.47}$$

In the supersymmetric phase with $W_k'\left(\chi_{\min} \cdot Z_k^{-1}\right) = 0$ the bosonic and fermionic mass is given by

$$m_k^2 = \frac{W_k''\left(\chi_{\min} \cdot Z_k^{-1}\right)^2}{Z_k^4} \neq 0. \tag{5.48}$$

$W_k''(\phi)$ stays positive for a typical flow as $k \to 0$ is approached. As k drops below the mass scale this leads to a decoupling of the massive modes.

In the LPA considered here we use $\eta = 0$ to calculate the masses across the phase transition. In figure 5.6 we show the mass depending on the relevant direction $\bar{\lambda}_\Lambda \bar{a}_\Lambda^2$ with $\bar{\lambda}_\Lambda = 0.1$. At $\bar{\lambda}_\Lambda \bar{a}_\Lambda^2|_{\text{crit}} \Lambda^{-1} \simeq 0.045$ the phase transition from the phase with unbroken to the phase with broken supersymmetry occurs. At this point both the fermionic and the bosonic mass are zero. In the broken phase the Goldstino is massless and the bosonic mass remains massive at a non-vanishing k and goes to zero for $k \to 0$.

6 The three-dimensional $\mathcal{N} = 1$ Wess-Zumino model

In this chapter we investigate the three-dimensional $\mathcal{N} = 1$ Wess-Zumino model. Many things in three dimensions are similar to those discussed in the previous chapter on the $\mathcal{N} = 1$ model in two dimensions. For this reason, we keep the derivation of the flow equations brief and mention only the differences to the two-dimensional case.

As the field is no longer dimensionless, the three-dimensional equations can be generalised to arbitrary dimensions straightforwardly.

As an application we also study the model at finite temperatures. Finite temperature introduces a supersymmetry breaking due to different statistics of fermions and bosons (cf. section 6.3). The derivation of the flow equations at finite temperatures is described in detail.

Three-dimensional supersymmetric scalar models at zero and finite temperature have previously been investigated by M. Moshe and coworkers [130, 131]. However they focused on supersymmetric $O(N)$ models in the limit of large N.

The construction of the superspace is similar to [132] where the $\mathcal{N} = 1$ superfield in three dimensions is introduced in the context of nonlinear sigma models.

In three-dimensional Euclidean space-time there exist no Majorana fermions. Due to this, three-dimensional Minkowski-space is considered here. To calculate the flow equations the integrals in the flow equation to Euclidean space time are Wick-rotated. The convention for the metric is $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1 - 1)$.

The results presented here are published in [133]. This chapter is organised as follows: After presenting the model and discussing the flow equations in the local potential approximation the fixed-point structure is investigated at leading and at next-to-leading order in the derivative expansion. The zero-temperature phase diagram is discussed as well as the behaviour of the bosonic mass in the SuSy broken phase. Then we derive the flow equations at finite temperature. SuSy breaking due to finite temperature is explicitly demonstrated. Also the pressure of a gas of scalar fields is calculated. We shall show that at finite temperature a broken $\mathbb{Z}_2$ symmetry, which is taken as a remnant of SuSy breaking, is always restored at some critical temperature.

6.1 The Wess-Zumino model

As in the two-dimensional model the system contains a bosonic field, an auxiliary field and a Majorana fermion. They are combined into a real superfield

$$\Phi(x,\theta) = \phi(x) + \bar{\theta}\psi(x) + \frac{1}{2}\bar{\theta}\theta F(x). \tag{6.1}$$

The supersymmetry transformations are generated by the supercharges:

$$\delta_\varepsilon \Phi = \mathrm{i}\bar{\varepsilon} Q\Phi, \quad Q = -\mathrm{i}\frac{\partial}{\partial\bar{\theta}} - (\gamma^\mu\theta)\partial_\mu\,. \tag{6.2}$$

For the γ matrices a Majorana representation is used:

$$\gamma^0 = \sigma_2,\ \gamma^1 = i\sigma_3 \text{ and } \gamma^2 = i\sigma_1. \tag{6.3}$$

With the aid of the symmetry relations for Majorana spinors

$$\bar{\psi}\chi = \bar{\chi}\psi,\ \bar{\psi}\gamma^\mu\chi = -\bar{\chi}\gamma^\mu\psi \tag{6.4}$$

and the particular Fierz identity $\theta\bar{\theta} = -\frac{1}{2}\bar{\theta}\theta\cdot\mathbb{1}$ the transformation laws for the component fields from equation (6.2) read

$$\delta\phi = \bar{\varepsilon}\psi, \quad \delta\psi = (F + \mathrm{i}\partial\!\!\!/\phi)\varepsilon, \quad \delta F = \mathrm{i}\bar{\varepsilon}\partial\!\!\!/\psi\,. \tag{6.5}$$

The anticommutator of two supercharges yields $\left\{Q_\alpha, \bar{Q}^\beta\right\} = 2(\gamma^\mu)_\alpha{}^\beta\partial_\mu$ and the supercovariant derivatives are

$$D = \frac{\partial}{\partial\bar{\theta}} + \mathrm{i}(\gamma^\mu\theta)\partial_\mu, \quad \text{and} \quad \bar{D} = -\frac{\partial}{\partial\theta} - \mathrm{i}(\bar{\theta}\gamma^\mu)\partial_\mu. \tag{6.6}$$

Moreover, we have

$$\{D_\alpha, \bar{D}^\beta\} = -2(\gamma)_\alpha{}^\beta\partial_\mu\,. \tag{6.7}$$

The off-shell Lagrangian is the $\bar{\theta}\theta$-component of $\bar{D}\Phi D\Phi + 2W(\Phi)$ and reads

$$\mathcal{L}_{\text{off}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}F^2 + FW'(\phi) - \frac{1}{2}W''(\phi)\bar{\psi}\psi. \tag{6.8}$$

By eliminating the auxiliary field with its equation of motion, $F = -W'(\phi)$, we obtain the on-shell Lagrangian density

$$\mathcal{L}_{\text{on}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}W'^2(\phi) - \frac{1}{2}W''(\phi)\bar{\psi}\psi. \tag{6.9}$$

From this expression it can be read off that $W'^2(\phi)$ acts as potential for the scalar fields.

6.2 The supersymmetric flow equations at zero temperature

Because Majorana fermions in three dimensions exist only in Minkowski space, a formulation of the Wetterich equation in Minkowski-space is needed. The derivation of such an equation can be found in appendix C following [134]. Formulations of the flow equation in Minkowski space can also be found in [135, 136]. The equation reads

$$\partial_t \Gamma_k = \frac{\mathrm{i}}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_t R_k\right], \quad t = \ln\left(\frac{k^2}{\Lambda^2}\right). \tag{6.10}$$

$\Gamma_k^{(2)}$ is defined as in the previous chapters. The supertrace is taken over Lorentz and internal indices as well as space-time or momentum coordinates. Here the 'RG-time' t is defined as $\ln\left(k^2/\Lambda^2\right)$. This is required in order to make a Lorentz invariant separation into large and small momenta.

Because the supersymmetric flow equations, including the choice of the regulator, are constructed along the lines of the previous chapters the construction will only be sketched here. The supersymmetric cutoff action is again quadratic in the superfields and contains a function of $D\bar{D}$. With the help of the anticommutation relations, powers of $D\bar{D}$ can always be decomposed into

$$\left(\frac{1}{2}D\bar{D}\right)^{2n} = (-\Box^n), \tag{6.11}$$

such that the cutoff action takes the form

$$\Delta S_k = \int \mathrm{d}\bar{\theta}\mathrm{d}\theta\mathrm{d}^3x\Phi\left(r_1(k,\Box) + r_2(k,\Box)\frac{1}{2}D\bar{D}\right)\Phi.$$

This leads to the same regulator structure that was used in the previous chapters. The conventions for the Fourier transformation are $i\partial_\mu \to q_\mu$. Then the bosonic and fermionic part of the regulator read

$$R_k^{\mathrm{B}} = \begin{pmatrix} q^2 r_2\left(k,q^2\right) & r_1\left(k,q^2\right) \\ r_1\left(k,q^2\right) & r_2\left(k,q^2\right) \end{pmatrix} \quad \text{and} \quad R_k^{\mathrm{F}} = -r_1\left(k,q^2\right) - r_2\left(k,q^2\right)\not{q}\,. \tag{6.12}$$

The explicit calculation can be found in [133].

6.2.1 The local potential approximation

In the LPA the ansatz for the effective action is

$$\Gamma_k = \int d^3x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\mathrm{i}}{2}\bar{\psi}\not{\partial}\psi + \frac{1}{2}F^2 + FW_k'(\phi) - \frac{1}{2}W_k''(\phi)\bar{\psi}\psi\right]. \tag{6.13}$$

Projecting equation (6.10) onto the terms linear in the auxiliary field and integrating with respect to ϕ yields the flow equation for the superpotential. Performing a Wick rotation of the zeroth component of the momentum, i. e. $q_0^M \to \mathrm{i} q_0^E$, the flow equation takes the form

$$\partial_k W_k(\phi) = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{\partial_k r_1(1 + r_2) - \partial_k r_2(W_k''(\phi) + r_1)}{q^2(r_2 + 1)^2 + (W_k''(\phi) + r_1)^2} \,. \tag{6.14}$$

Formally the flow equation is the same in two and three dimensions, only the integration measure changes.

In the following the simple regulator functions in Euclidean space time,

$$r_1 = 0, \quad r_2 = \left(\left| \frac{k}{q} \right| - 1 \right) \theta \left(k^2 - q^2 \right), \tag{6.15}$$

is chosen for which the momentum integration in (6.14) can be performed analytically. Contrary to the model in two dimensions, the regulator function (6.15) regularises the flow even if a running wave-function renormalisation is taken into account. For the superpotential $W_k(\phi)$ the flow equation simplifies to

$$\partial_k W_k(\phi) = -\frac{k^2}{8\pi^2} \frac{W_k''(\phi)}{k^2 + W_k''(\phi)^2}. \tag{6.16}$$

As we are interested in the effective potential for the scalar field $V(\phi) = \lim_{k\to 0} \frac{1}{2} W_k'^2(\phi)$, we consider its flow equation:

$$\partial_k W_k' = -\frac{k^2 W_k^{(3)}(\phi) \left(k^2 - W_k''(\phi)^2 \right)}{8\pi^2 \left(k^2 + W_k''(\phi)^2 \right)^2}. \tag{6.17}$$

Figure 6.1 shows the flow of $W_k'^2(\phi)$ for a cubic superpotential at the ultraviolet cutoff scale, $W_\Lambda' = \lambda_\Lambda(\frac{1}{3}\phi^3 - a_\Lambda^2 \phi)$, and with initial conditions $\lambda_\Lambda \Lambda^{-1} = 1$, $a_\Lambda^2 \Lambda^{-1/2} = 0.02$. With these initial conditions the RG flow starts in the regime with broken $\mathbb{Z}_2$ symmetry and for $k \to 0$ ends up in the regime with restored $\mathbb{Z}_2$ symmetry. The potential $W_k(\phi)$ becomes flat at the origin as k is lowered. In three dimensions, however, the function $W_{k\to 0}''(\phi)$ is regular for all values of the field, in contrast to the situation in two dimensions.

As in two dimensions we first investigate the fixed-point structure. In order to do this, we introduce dimensionless quantities

$$\varphi = k^{-1/2}\phi, \; w_t(\varphi) = k^{-2} W_k(\phi), \; w_t'(\varphi) = k^{-3/2} W_k'(\phi), \; \dots . \tag{6.18}$$

The dimensionless flow equation for the superpotential then reads

$$\partial_t w_t + 2 w_t - \frac{\varphi w_t'}{2} = -\frac{w_t''}{8\pi^2(1 + w_t''^2)} \,, \tag{6.19}$$

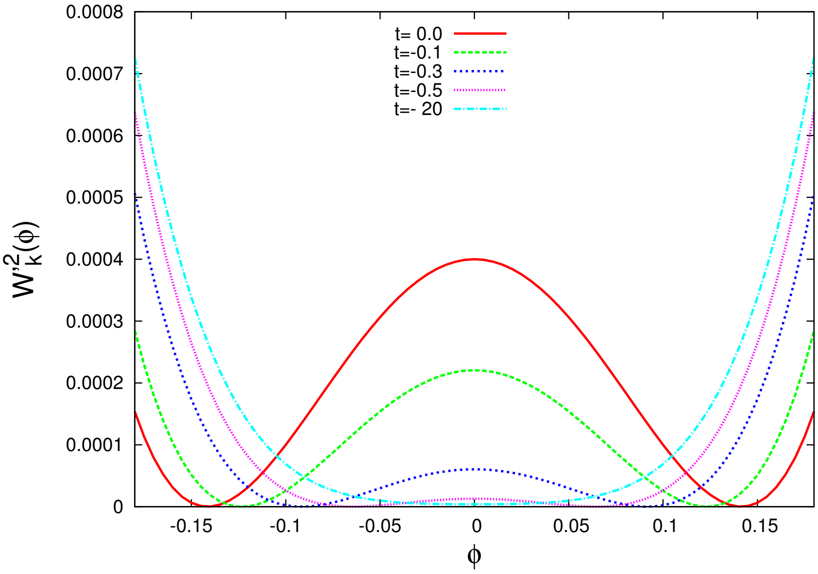


Figure 6.1: RG flow of $W_k'^2(\phi)$ with the initial conditions $\lambda_\Lambda \Lambda^{-1} = 1$, $a_\Lambda^2 \Lambda^{-1/2} = 0.02$.

where the prime denotes the derivative with respect to the dimensionless field φ. Its fixed points are characterised by $\partial_t w_* = 0$. In contrast to the case in two dimensions, there appears now the additional term $\propto \varphi w_t'(\varphi)$, since the field ϕ itself is a dimensionful quantity. In d dimensions the dimensionless flow equation with the same regulator generalises to

$$\partial_t w_t + (d-1)w_t - (d-2)\frac{\varphi w_t'}{2} = -\frac{2^{-d}\pi^{-d/2}}{(d-1)\Gamma(d/2)}\frac{w_t''}{(1+w_t''^2)}\,. \tag{6.20}$$

As already stated in two dimensions, it follows that the couplings of the terms φ^0 and φ^1 do not enter the fixed-point equation but evolve independently. This is due to the fact that in the supersymmetric theory it is always possible to make a polynomial expansion around $\varphi = 0$, that is the minimum of w', *even* if the bosonic potential V at the cutoff scale is a double-well potential. In bosonic $O(N)$ models the expansion point for a double-well potential is the minimum of the potential which lies at a point $\varphi \neq 0$. By this the coupling at lowest order enters in the flow equations of the higher order couplings for the bosonic theories.

This has some interesting consequences which distinguish the supersymmetric Wess-Zumino model from purely bosonic theories, as for example $O(N)$ models in three dimensions, see e. g. [17, 137, 138, 139] for results on these models. These consequences are discussed below.

For the fixed-point analysis, we need the first derivative of equation (6.19),

$$\partial_t w_k' + \frac{3w_k' - \varphi w_k''}{2} = \frac{(w_k''^2 - 1)w_k'''}{8\pi^2(1+w_k''^2)^2}. \tag{6.21}$$

As in two dimensions, we first consider a polynomial approximation of the flow equation. The

	$r_2 = \left({}^{k}/_{q} - 1\right) \theta\left({}^{k^2}/_{q^2} - 1\right)$						$r_2 = \left({}^{k^2}/_{q^2} - 1\right) \theta\left({}^{k^2}/_{q^2} - 1\right)$					
$2n$	$\pm\lambda$	$\pm b_4$	$\pm b_6$	$\pm b_8$	$\pm b_{10}$	$\pm b_{12}$	$\pm\lambda$	$\pm b_4$	$\pm b_6$	$\pm b_8$	$\pm b_{10}$	$\pm b_{12}$
4	1.546	2.305					1.952	3.491				
6	1.590	2.808	6.286				2.013	4.256	9.407			
8	1.595	2.873	7.150	13.41			2.019	4.329	10.37	14.68		
10	1.595	2.873	7.155	13.48	1.212		2.018	4.319	10.23	12.64	-35.12	
12	1.595	2.870	7.118	12.90	-8.895	-183.3	2.018	4.313	10.16	11.43	-56.14	-380.4

Table 6.1: Wilson-Fisher fixed point as obtained from the polynomial approximation of $w'(\varphi)$ with two different regulators.

expansion of the superpotential reads

$$w'_k(\varphi) = \lambda(t)\left(\varphi^2 - a^2(t)\right) + \sum_{i=2}^{n} b_{2i}(t)\,\varphi^{2i}\,. \tag{6.22}$$

This yields the system of coupled differential equations:

$$\begin{aligned}
\partial_t a^2(t) &= a^2(t)\left(-\frac{3\lambda(t)^2}{\pi^2} + \frac{3b_4(t)}{2\pi^2\lambda(t)} - 1\right) + \frac{1}{4\pi^2}\,,\\
\partial_t \lambda(t) &= -\frac{3b_4(t) - 6\lambda(t)^3 + \pi^2\lambda(t)}{2\pi^2}\,,\\
\partial_t b_4(t) &= \frac{120b_4(t)\lambda(t)^2 + 2\pi^2 b_4(t) - 15b_6(t) - 80\lambda(t)^5}{4\pi^2}\\
&\dots
\end{aligned} \tag{6.23}$$

Determining the fixed point solutions from this system yields an ultraviolet-stable Gaußian fixed point with all couplings equal to zero and a pair of nontrivial maximally infrared-stable fixed points which are related by a $\mathbb{Z}_2$ symmetry. They are regarded as one fixed point in the following. The nontrivial fixed point turns out to be the supersymmetric analogue of the Wilson-Fisher fixed point in bosonic theories. We find no other solutions to the fixed-point equations.

As the maximally infrared-stable fixed point in two dimensions, the Wilson-Fisher fixed point has one infrared unstable direction, namely the coupling a_t^2. Compared to the two-dimensional maximally infrared-stable fixed point, the convergence of the fixed-point couplings with the order of the truncation is faster in three dimensions. The fixed-point values for the couplings with increasing truncation are shown in table 6.1. As the unstable direction does not feed back into the equation for the higher-order couplings they always flow into the Wilson-Fisher fixed-point without fine tuning.

The critical exponents for the Wilson-Fisher fixed point are obtained along the same lines as in two dimensions. The critical exponent for the infrared-unstable direction takes the value

	$2n$	critical exponents								
$\left(k/\lvert q\rvert - 1\right)$	6	-0.799	-5.92	-20.9						
	8	-0.767	-4.83	-14.4	-38.2					
	10	-0.757	-4.35	-11.5	-26.9	-60.8				
	12	-0.756	-4.16	-9.94	-21.4	-43.8	-89.0			
	14	-0.756	-4.10	-9.13	-18.3	-35.1	-65.4	-123		
	16	-0.756	- 4.08	-8.72	-16.4	-29.9	-52.9	-91.9	-163	
	18	-0.756	- 4.08	-8.54	-15.2	-26.4	-45.0	-75.0	-124	-209
$\left(k^2/q^2 - 1\right)$	6	-0.770	-6.02	-22.6						
	8	-0.732	-4.74	-14.8	-41.5					
	10	-0.723	-4.19	-11.4	-28.0	-66.3				
	12	-0.722	-3.98	-9.67	-21.6	-46.1	-97.4			
	14	-0.722	-3.92	-8.76	-18.1	-35.9	-69.3	-134		
	16	-0.723	-3.90	-8.31	-15.9	-29.9	-54.6	-97.9	-179	
	18	-0.723	-3.91	-8.11	-14.6	-26.0	-45.6	-78.0	-132	-229

Table 6.2: Critical exponents for the Wilson-Fisher fixed point for different truncations and two different regulators $\left(k/\lvert q\rvert - 1\right)\theta\left(k^2/q^2 - 1\right)$ and $\left(k^2/q^2 - 1\right)\theta\left(k^2/q^2 - 1\right)$.

$\nu^{-1} = 3/2$. The other critical exponents are listed in table 6.2 for different truncations and two different regulators

$$r_2 = \left(\frac{k}{\lvert q\rvert} - 1\right)\theta\left(\frac{k^2}{q^2} - 1\right) \quad \text{and} \quad r_2 = \left(\frac{k^2}{q^2} - 1\right)\theta\left(\frac{k^2}{q^2} - 1\right). \tag{6.24}$$

Now we solve the partial differential equation for the fixed point potential. As in two dimensions the infrared-stable solution is found if the second derivative of equation (6.19) is considered. For simplicity of notation again $w''(\varphi) = u$ is introduced and the fixed-point equation for the second derivative reads

$$u''(u^2 - 1) = 2u\,\frac{u^2 - 3}{u^2 + 1}\,u'^2 + 4\pi^2(u^2 + 1)^2(2u - \varphi u'). \tag{6.25}$$

This equation has the asymptotic solution $u_{\text{as}} \sim \varphi^2$. Again, the term $(u^2 - 1)$ arises due to the sign change in equation (6.21). As in two dimensions, the condition for having a regular solution at $u^2 = 1$ leads to the condition that the right-hand side has to vanish at this point. This leads to a condition

$$u'(\varphi_{\text{crit}}) = 4\left(\mp\varphi_{\text{crit}}\pi^2 - \sqrt{\pi^2 + \varphi_{\text{crit}}^2\pi^4}\right) \tag{6.26}$$

on the slope at the critical point.

Solving the equation with Mathematica 7 yields an odd and regular solution with the starting conditions $u(0) = 0$ and $u'(0) = 2\lambda = \pm 2 \cdot 1.59508$. The Taylor expansion of this solution around zero corresponds to the polynomial solution discussed above.

From the asymptotic behaviour of the dimensionless potential follows for the asymptotic behaviour of the dimensionful potential

$$w''(\varphi \to \pm\infty) \simeq \pm\varphi^2 \Rightarrow W''_*(\phi \to \pm\infty) \simeq \pm\phi^2.$$

The bosonic potential that is derived from this superpotential behaves asymptotically as $V \sim \phi^6$. It is therefore justified to call this fixed point the supersymmetric analogue of the Wilson-Fisher fixed point in three-dimensional $O(N)$ theories.

After we have established the fixed point structure at the order of the LPA now we investigate the next-to-leading order.

6.2.2 Next-to-leading order

At next-to-leading order the ansatz for the effective action reads

$$\Gamma_k = \int d^3x \left(\frac{1}{2} Z_k^2 \left(\partial_\mu\phi\partial^\mu\phi - \mathrm{i}\bar{\psi}\partial\!\!\!/\psi + F^2 \right) + F W_k'(\phi) - \frac{1}{2} W_k''(\phi)\bar{\psi}\psi \right) . \qquad (6.27)$$

Again, we consider the simplest ansatz at NLO and neglect a field- and momentum dependence of Z_k. The anomalous dimension η stays small compared to one. The flow equation of the superpotential is obtained from a projection on the part linear in the auxiliary field. The flow equation for the wave-function renormalisation follows from the projection on the parts quadratic in the auxiliary field. Employing the same regulator as before the flow equations are

$$\partial_k W_k(\phi) = -\frac{k^2 W_k''(\phi)}{24\pi^2} \frac{k\partial_k Z_k^2 + 3Z_k^2}{k^2 Z_k^4 + W_k''(\phi)^2} , \qquad (6.28)$$

$$\partial_k Z_k^2 = -\frac{k^2}{4\pi^2} (k\partial_k Z_k^2 + 2Z_k^2) \frac{Z_k^2 W_k^{(3)}(\phi)^2 \left(k^2 Z_k^4 - W_k''(\phi)^2\right)}{\left(k^2 Z_k^4 + W_k''(\phi)^2\right)^3} \Bigg|_{\phi=0} . \qquad (6.29)$$

As the wave-function renormalisation is independent of the fields, equation (6.29) can be projected on $\phi = 0$. Rescaling the fields with the canonical dimension and the wave-function renormalisation,

$$\chi = Z_k k^{-1/2} \phi, \quad \mathfrak{w}(\chi) = k^{-2} W_k(\phi), \qquad (6.30)$$

the dimensionless flow equations read

$$\partial_t \mathfrak{w} + 2\mathfrak{w} - \frac{1}{2}(1+\eta)\chi\,\mathfrak{w}' = -\frac{(3-\eta)\mathfrak{w}''}{24\pi^2(1+\mathfrak{w}''^2)} , \quad \eta = \frac{(2-\eta)(1-\mathfrak{w}''^2)\mathfrak{w}'''^2}{4\pi^2(1+\mathfrak{w}''^2)^3}\Bigg|_{\chi=0} . \qquad (6.31)$$

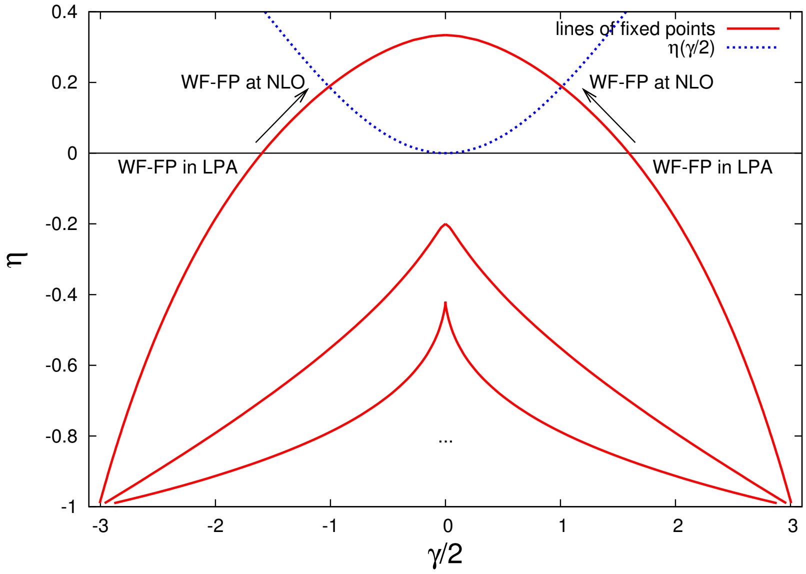


Figure 6.2: Lines of fixed points in the η-γ plane (solid curves) and the anomalous dimension as a function of $\gamma = 2\lambda$ as obtained from equation (6.31) (dotted curve). The Wilson-Fisher fixed point (WF-FP) is given by the intersection of these curves.

As expected, the structure of the flow equation at leading order and at next-to-leading order are very similar.

For the discussion of the fixed-point structure, we consider η as a free parameter as in the previous chapter. Again, lines of fixed points emerge as in two dimensions. This is shown in figure 6.2. The pictures in two and three dimensions are very similar. Indeed, they would be identical but for a shift if the regulators were the same. However, in three dimensions the lines of fixed points are shifted to lower η values. For $\eta = 0$ this results in just *two* fixed points which are related by the $\mathbb{Z}_2$ symmetry. For $\eta \neq 0$ only the couplings change but not the fact that only two fixed points exist.

As in two dimensions, a superscaling relation between the critical exponent of the infrared-unstable direction and the anomalous dimension can be derived from a polynomial expansion of the fixed point equations. The flow equation for the coupling a_t^2 reads

$$\partial_t a_t^2 = -\frac{a_t^2}{\lambda_t}\partial_t\lambda_t + \frac{\eta-3}{12\pi^2} + \frac{\eta-3}{2}a_t^2. \tag{6.32}$$

From this, we can read off the superscaling relation

$$\nu_W^{-1} = \frac{3-\eta}{2}. \tag{6.33}$$

The truncation dependence of the anomalous dimension is smaller than in two dimensions, cf. table 6.3.

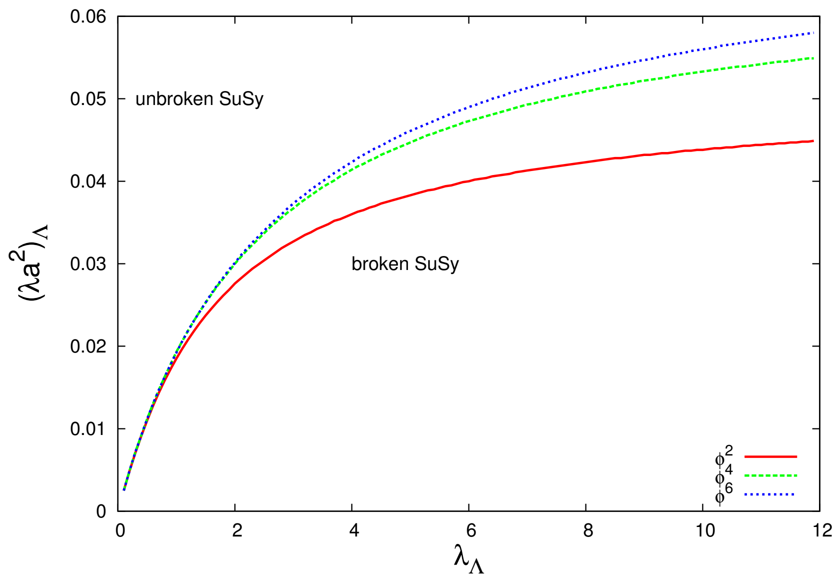


Figure 6.3: Phase diagram in the plane spanned by the dimensionless couplings specified at the cutoff scale Λ as obtained from truncations with $n = 1$ (ϕ^2), $n = 2$ (ϕ^4), and $n = 3$ (ϕ^6) in equation (6.22).

$2n$	4	6	8	10	12	14
η_*	0.187711	0.188258	0.18802	0.187996	0.188001	0.188003

Table 6.3: Dependence of the fixed-point value of the anomalous dimension η on the truncation

6.2.3 Phase diagram and the scaling of the mass

We calculate the phase diagram in the local potential approximation. The qualitative behaviour stays the same at next-to-leading order, only the quantitative values change. For the phase diagram shown in figure 6.3 the same picture as in two dimensions emerges. The critical point is reached by fine-tuning the infrared-unstable direction $a_t^2 \sim k^{-3/2}$. Again, there is a maximal value for $\bar{\lambda}_\Lambda \bar{a}_\Lambda^2$ above which supersymmetry cannot be broken dynamically. Keep in mind, however, that the values $\bar{\lambda}_\Lambda$ and $\bar{a}_\Lambda^2$ are not universal quantities and therefore regulator dependent.

In the broken phase, the minimum of the bosonic potential is at $\phi = 0$ and therefore a polynomial expansion around this minimum is justified. This implies that the mass in the broken regime with $W_k''(\phi_{\text{min}} = 0) = 0$ is given by

$$m^2(k) = W_k'(\phi_{\text{min}} = 0)\, W_k'''(\phi_{\text{min}} = 0) = 2k^2\lambda^2 a^2 \sim k^{1/2}. \qquad (6.34)$$

In figure 6.4 the logarithm of the bosonic mass in the broken regime is displayed as a function of the RG scale k. From a linear fit it follows that $m(k) \sim k^{0.23}$ for $k \ll \Lambda$ which is reasonable close to the prognosticated scaling behaviour $m(k) \sim k^{1/4}$.

The theory flows into the massless conformal limit because the unstable direction does not feed

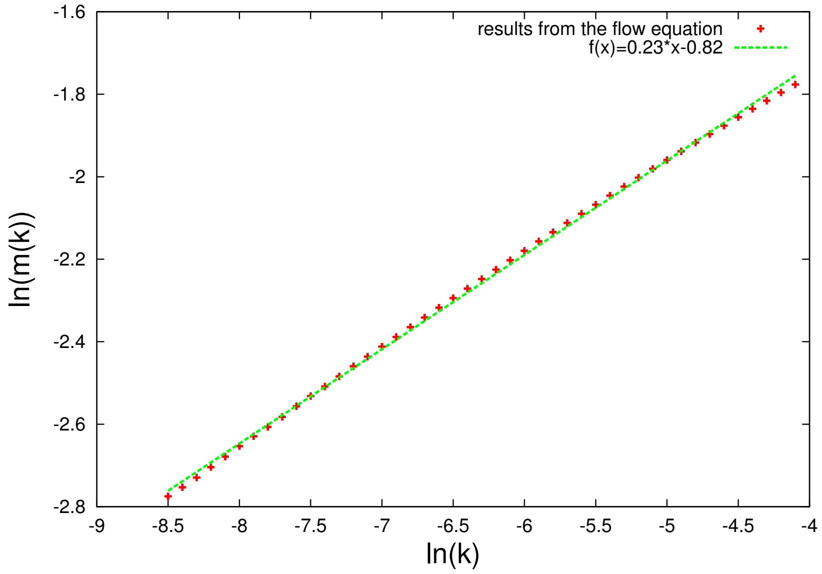


Figure 6.4: Logarithm of the boson mass as a function of the RG scale k. A linear fit to the data points yields $m(k) \sim k^{0.23}$ for $k \ll \Lambda$.

back into the other equations and therefore the second derivative of the superpotential always flows into its infrared-stable fixed point. This is different to the behaviour known from $O(N)$ models with finite N, where the unstable direction induces the non-vanishing mass and makes it necessary to fine-tune the ultraviolet parameters. However, in the large-N limit, the vacuum expectation value of the field, which corresponds to a_t^2 here, decouples from the flow equations of the higher-order couplings at least to low order in the polynomial expansion [137].

6.3 Finite-temperature flow equations

We restrict the discussion of the model at finite temperature to the LPA. This approximation should be sufficient to capture at least the qualitative features of this model, see e. g. [140].

Supersymmetry at finite temperature has been investigated extensively in the literature, see e. g. [141, 142, 143]. In contrast to most other symmetries which are broken at low temperature and restored at finite temperature SuSy is broken at any finite temperature.

The reason for this is that bosons and fermions have different statistics and therefore are treated differently by the heat bath at finite temperatures. They are no longer related as they are for unbroken supersymmetry. This is often referred to as *soft SuSy breaking*. In [141] it is argued that the breaking due to the interaction with the heat bath is spontaneous and they find a massless Goldstone fermion associated to the breaking.

In this section the SuSy breaking caused by finite temperature is studied. To discuss a phase diagram even at finite temperature we use a remnant of SuSy breaking, the restoration of $\mathbb{Z}_2$ breaking, which still occurs at finite temperature.

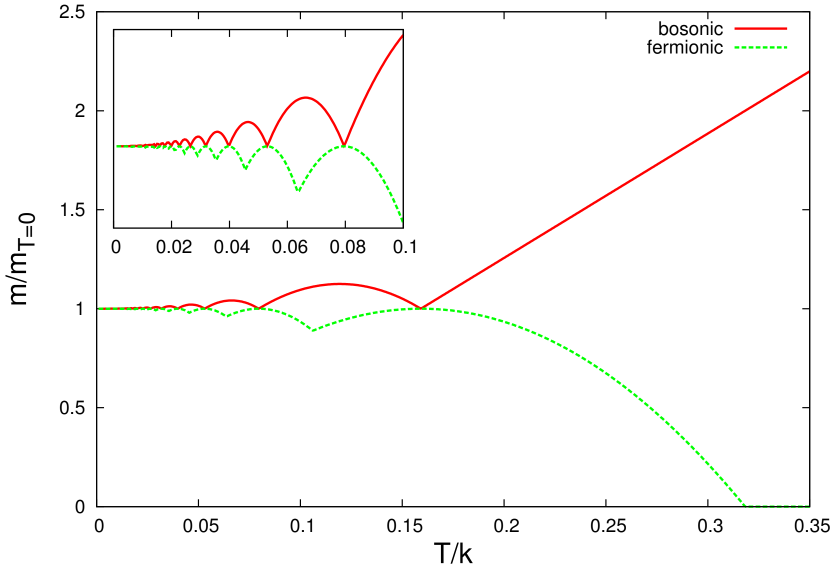


Figure 6.5: The temperature dependent masses for bosons and fermions

The flow equations in the finite temperature case are obtained from the flow equations at zero momentum by replacing the time-like momentum integration by a sum over Matsubara frequencies

$$p_0 \longrightarrow \left\{ \begin{array}{c} \omega_n \\ \nu_n \end{array} \right\}, \; n = 0, 1, \dots \qquad \int \frac{dp_0}{2\pi} \dots \longrightarrow T \sum_{n=-\infty}^{\infty} \dots , \tag{6.35}$$

where $\omega_n = 2\pi n T$ denotes the bosonic frequencies and $\nu_n = (2n+1)\pi T$ the fermionic ones. Similar sums have been investigated in previous works on finite-temperature FRG [144, 145, 146].

The derivation of the finite temperature flow equation can be found in appendix D. For the simple regulator

$$r_1 = 0 \quad \text{and} \quad r_2 = \left(\left| \frac{k}{q} \right| - 1 \right) \theta \left(\frac{k^2}{p^2} - 1 \right), \tag{6.36}$$

the Matsubara sums can be calculated analytically and the flow equations read

$$\partial_k W_k'^{\,\text{bos}} = -\frac{k^2}{8\pi^2} W_k''' \frac{k^2 - W_k''^2}{(k^2 + W_k''^2)^2} \left(\frac{\pi T}{k} - (2s_{\text{B}} + 1)^2 \frac{\pi T}{k} + 2(2s_{\text{B}} + 1) \right) \frac{\pi T}{k}, \tag{6.37}$$

$$\partial_k W_k'^{\,\text{ferm}} = -\frac{k^2}{8\pi^2} \frac{(k^2 - W_k''^2) W_k'''}{(k^2 + W_k''^2)^2} \left(1 - \left(1 - \frac{2 s_{\text{F}} \pi T}{k} \right)^2 \right), \tag{6.38}$$

where the *temperature-dependent* floor-functions s_{B} and s_{F} are given by

$$s_{\text{B}} = \left\lfloor \frac{k}{2\pi T} \right\rfloor \quad \text{and} \quad s_{\text{F}} = \left\lfloor \frac{k}{2\pi T} + \frac{1}{2} \right\rfloor . \tag{6.39}$$

At finite temperature the flow equations differ from the ones at zero temperature by an additional temperature-dependent factor. This factorisation in a temperature-dependent and a temperature-independent part is due to the regulator function [146].

The differences in the flow of the superpotentials arises from the supersymmetry breaking due to the different boundary conditions for fermions and bosons (cf. section 6.3). In the limit of $T \to 0$ the temperature-dependent functions reduce to one and both flow equations become the same again.

SuSy breaking can be observed for example in the different masses for bosons and fermions as the temperature is increased. We display these masses in figure 6.5. The picture found here is very similar to the one encountered in non-supersymmetric theories [146].

For $T/k > (2\pi)^{-1}$ the bosonic mass is proportional to the temperature. This is due to the bosonic $n = 0$ Matsubara mode which dominates in this temperature regime. The fermionic mass reaches zero at $T/k = \pi^{-1}$ because for fermions there exists no $n = 0$ Matsubara mode. The spikes are caused by the θ-function in the regulator which cuts off the n-th Matsubara mode at $T/k > (2\pi n)^{-1}$ for bosons and $T/k > \pi^{-1}(2n+1)^{-1}$ for fermions respectively.

6.3.1 Pressure

In the zero temperature case, the bosonic mass tends to zero in the phase with broken supersymmetry and restored $\mathbb{Z}_2$ symmetry. The system should therefore behave as a gas of massless bosons and obey a Stefan-Boltzmann law in 2 + 1 dimensions. From this it is inferred that the pressure should be given by

$$\Delta p = \frac{\zeta(3)}{2\pi} T^3. \tag{6.40}$$

In $O(N)$ symmetric theories at finite temperature the couplings consist of a temperature-independent and a temperature-dependent part. In contrast to the temperature-independent part the latter does not need to be renormalised. The temperature-independent part has to be removed and this is done by a subtraction of this part. The pressure is therefore defined as

$$-\Delta p = \left(V_T^{\text{eff}} - V_{T=0}^{\text{eff}} \right)_{\phi=\phi_{\min}}. \tag{6.41}$$

In the supersymmetric theory the subtraction has to be performed on the level of the couplings as well. From this it follows that the pressure is given by

$$\Delta p = \frac{1}{2} (W_k'|_{T=0} - W_k'|_T)^2_{\phi=\phi_{\min}}. \tag{6.42}$$

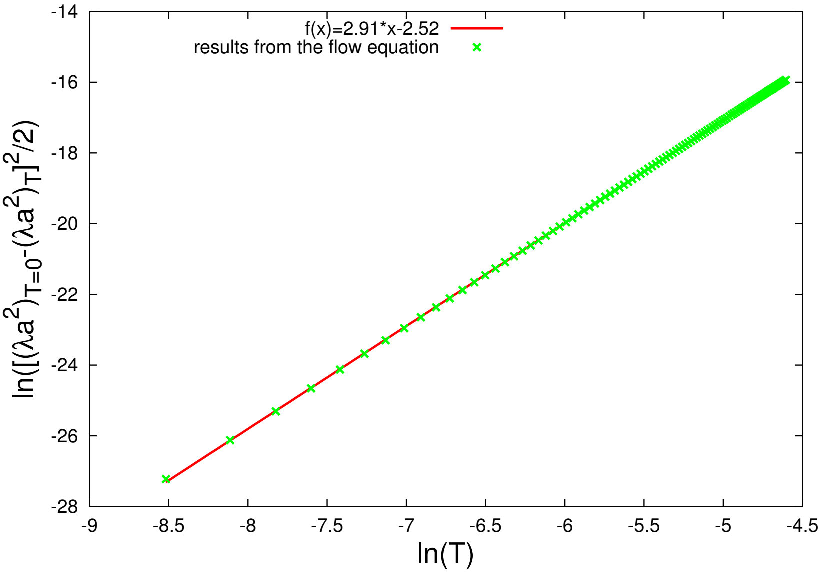


Figure 6.6: Double logarithmic plot of the pressure versus the temperature.

A numerical calculation in the phase with restored $\mathbb{Z}_2$ with the parameters

$$\lambda = 1,\ a^2 = -0.1, \quad T \in [10^{-4}, 10^{-2}] \tag{6.43}$$

yields

$$\frac{\left[(\lambda a^2)_{T=0} - (\lambda a^2)_{T\neq 0}\right]^2}{2} = 0.08 \cdot T^{2.91}. \tag{6.44}$$

This is shown in figure 6.6. The power law behaviour is compatible with the one expected from the Stefan-Boltzman law, whereas there are deviations in the prefactor. Possible reasons for this are that the $k = 0$ limit has not sufficiently been reached and therefore the boson is not truly massless or that self-interactions are present which lead to a deviation from the ideal Bose gas limit [147, 148].

6.3.2 High-temperature expansion and dimensional reduction

The model displays some interesting features in the high temperature limit $T \gg k$. In this case, the floor functions vanish and the flow equations reduce to

$$\partial_k W_k'^{\,\mathrm{bos}} = -\frac{k^2}{8\pi^2} W_k''' \frac{k^2 - W_k''^2}{(k^2 + W_k''^2)^2} \frac{2\pi T}{k} \quad \text{and} \quad \partial_k W_k'^{\,\mathrm{ferm}} = 0\,. \tag{6.45}$$

As suggested in [149] the bosonic flow equation can be rescaled with $\phi = \sqrt{T}\tilde{\phi}$ and $W_k(\phi) = T\tilde{W}_k(\tilde{\phi})$. This yields the two-dimensional flow equation. As expected, the model shows

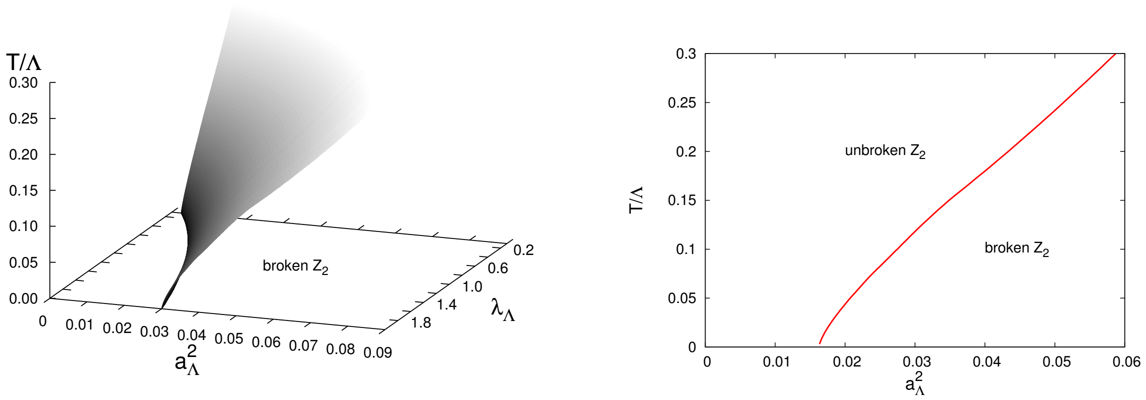


Figure 6.7: Finite-temperature phase diagram of the $\mathcal{N} = 1$ Wess-Zumino model.
Left panel: $\mathbb{Z}_2$ phase boundary in the space spanned by temperature T/Λ and the value of the couplings $(\lambda_\Lambda, a^2_\Lambda)$ at vanishing temperature. *Right panel:* Slice of the $\mathbb{Z}_2$ phase-boundary for fixed $\lambda_\Lambda = 0.8$.

dimensional reduction. However, the theory obtained in this limit is *not* supersymmetric because the fermions have dropped out of the flow due to the absence of a thermal zero mode.

The fixed-point couplings are rescaled with powers of T/k according to their canonical dimension and they show the following behaviour for $T/k \ll 1$:

$$^{3D}(a^2)_T = {}^{2D}(a^2)_{T=0} \left(\frac{T}{k}\right)^{1/2}, \quad {}^{3D}\lambda_T \ = {}^{2D}\lambda_{T=0} \left(\frac{T}{k}\right)^{-1/2}, \quad {}^{3D}(b_{2i})_T = {}^{2D}(b_{2i})_{T=0} \left(\frac{T}{k}\right)^{1/2-i}. \tag{6.46}$$

where ${}^{2D}(a^2)_{T=0}$, ${}^{2D}\lambda_{T=0}$ and ${}^{2D}(b_{2i})_{T=0}$ denote the fixed-point values of the couplings of the two-dimensional theory.

6.3.3 Phase diagram at finite temperature

At finite temperature supersymmetry is necessarily broken due to the different boundary conditions for bosons and fermions. However, the $\mathbb{Z}_2$ symmetry remains and is taken as a remnant of supersymmetry breaking in order to discuss the phase diagram concerning the breaking of this symmetry. As an order parameter we take again the sign change of $a^2_{k\to 0}$. Thus the case with broken $\mathbb{Z}_2$ symmetry of the ground state and soft supersymmetry breaking due to the boundary conditions and the case with unbroken $\mathbb{Z}_2$ symmetry of the ground state have to be distinguished.

Again, a truncation at ϕ^8 is used. The phase diagram is shown in figure 6.7, left panel. It is spanned by the temperature T measured in units of the cutoff and the couplings $(\lambda_\Lambda, a^2_\Lambda)$ specified at the cutoff scale at $T = 0$. As the initial conditions are specified at $T = 0$, the values in the phase diagram have to be restricted to temperatures that are small compared to the ultraviolet scale. In the $T = 0$ plane the phase transition line corresponds to the phase diagram in figure 6.3 (right panel) which separates the phase with unbroken supersymmetry from the one with broken supersymmetry. For couplings that are chosen such that at $T = 0$ the system is in a state with unbroken supersymmetry (and therefore broken $\mathbb{Z}_2$ symmetry), there is always a phase-transition temperature at which the $\mathbb{Z}_2$ symmetry of the ground state is restored.

In the right panel a slice through the phase diagram for a fixed coupling $\lambda_\Lambda = 0.8$ is shown. We observe that the phase-transition temperature increases as the coupling $(a_\Lambda^2)_{T=0}$ grows. On the other hand, an increase of the zero temperature coupling $(a_\Lambda^2)_{T=0}$ at the cutoff scale corresponds to an increase in the renormalised zero temperature coupling $(a_{k\to 0}^2)_{T=0}$. From this we conclude that an increase in the renormalised coupling at zero temperature leads to an increase in the phase-transition temperature. This is to be expected because in a $O(1) \simeq \mathbb{Z}_2$ theory the minimum of the bosonic potential $(\lambda_* a_{k\to 0}^2)_{T=0}$ sets the scale at $T = 0$. It therefore plays a role similar to the finite expectation value of the field in $O(N)$ models [150].

7 The two-dimensional $\mathcal{N} = (2,2)$ Wess-Zumino model

In this chapter we discuss the application of the FRG to the $\mathcal{N} = (2,2)$ Wess-Zumino model in two dimensions. For this model there exist results from Monte Carlo simulations on the lattice which can be used for a comparison [151]. In contrast to the previous models we consider here a *momentum-dependent* wave-function renormalisation.

To deal with flow equations that have the full momentum dependence, a numerical toolbox called FlowPy has been developed in cooperation with T. Fischbacher (Uni Southampton). This toolbox is designed to handle such flow equations.

The $\mathcal{N} = (2,2)$ Wess-Zumino model is studied intensively in the literature, see e. g. [120, 151, 152, 153, 154] for lattice simulations. Quantities that are investigated on the lattice are the renormalised mass or Ward identities. In contrast to the lattice calculations, Ward identities in the FRG approach are always fulfilled because the formalism is manifestly supersymmetric. Our focus lies on the renormalised masses instead.

The results presented in this chapter are published in [155]. This chapter is organised as follows: First the model is presented and the supersymmetric flow equations are derived. The flow equation for the superpotential yields directly the non-renormalisation theorem. As the superpotential is not renormalised, all renormalisation is carried by the wave-function renormalisation and the flow equation for this quantity – with full momentum dependence – is derived. The renormalised mass is then calculated and compared to the results on the lattice.

7.1 Description of the model

The $\mathcal{N} = (2,2)$ Wess-Zumino model in two dimensions is derived by a dimensional reduction of the $\mathcal{N} = 1$ model in four dimensions which was the original model introduced by Wess and Zumino [156].

The Lagrange density reads[1]

$$\mathcal{L}_{\text{off}} = 2\bar{\partial}\bar{\phi}\partial\phi + \bar{\psi}M\psi - \frac{1}{2}\bar{F}F + \frac{1}{2}FW'(\phi) + \frac{1}{2}\bar{F}\,\bar{W}'(\bar{\phi}), \tag{7.1}$$

[1]For a superspace formulation of this model see appendix E.1.

with Dirac fermions ψ and $\bar{\psi}$ and the fermion matrix

$$M = \bar{\partial} + W''(\phi)P_+ + \bar{W}''(\bar{\phi})P_-, \tag{7.2}$$

with the projectors $P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_*)$. The model contains complex auxiliary and scalar fields $F = F_1 + \mathrm{i}F_2$ and $\phi = \phi_1 + \mathrm{i}\phi_2$.

A suitable description is given by complex coordinates

$$z = x_1 + \mathrm{i}x_2, \quad \bar{z} = x_1 - \mathrm{i}x_2, \quad \partial = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2), \quad \bar{\partial} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2). \tag{7.3}$$

The superpotential is denoted by

$$W(\phi) = u(\phi_1, \phi_2) + \mathrm{i}v(\phi_1, \phi_2). \tag{7.4}$$

For the γ matrices we use the Weyl basis with $\gamma^1 = \sigma_1, \gamma^2 = -\sigma_2$ and $\gamma_* = \mathrm{i}\gamma^1\gamma^2 = \sigma_3$.

The complex spinors can be decomposed as $\psi = \begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}^{\mathrm{T}}$ and $\bar{\psi} = \begin{pmatrix}\bar{\psi}_1 & \bar{\psi}_2\end{pmatrix}$. The action is invariant under the supersymmetry transformations

$$\begin{aligned}
&\delta\phi = \bar{\psi}_1\varepsilon_1 + \bar{\varepsilon}_1\psi_1,\ \delta\bar{\psi}_1 = -\frac{1}{2}F\bar{\varepsilon}_1 - \partial\phi\bar{\varepsilon}_2,\ \delta\bar{\psi}_2 = -\bar{\partial}\bar{\phi}\bar{\varepsilon}_1 - \frac{1}{2}\bar{F}\bar{\varepsilon}_2, \delta F = 2(\partial\bar{\psi}_1\varepsilon_2 - \bar{\varepsilon}_2\bar{\partial}\psi_1),\\
&\delta\bar{\phi} = \bar{\psi}_2\varepsilon_2 + \bar{\varepsilon}_2\psi_2,\ \delta\psi_1 = -\frac{1}{2}F\varepsilon_1 + \bar{\partial}\phi\varepsilon_2,\ \delta\psi_2 = \partial\bar{\phi}\varepsilon_1 - \frac{1}{2}\bar{F}\varepsilon_2,\ \ \delta\bar{F} = 2(\partial\bar{\psi}_2\varepsilon_1 - \bar{\varepsilon}_1\bar{\partial}\psi_2).
\end{aligned} \tag{7.5}$$

$\varepsilon_{1,2}$ and $\bar{\varepsilon}_{1,2}$ are four real anticommuting parameters. Therefore the SuSy algebra is formed by four real supercharges. The algebra can be decomposed into a chiral (left-moving) and anti-chiral (right-moving) part. This is the reason for the notation $\mathcal{N} = (2,2)$.

Starting out from the SuSy transformations the superspace formulation of this model is constructed in appendix E.1. In this appendix also the most general action with Kähler potential is discussed.

Integrating out the auxiliary fields yields the on-shell Lagrangian

$$\mathcal{L}_{\mathrm{on}} = 2\bar{\partial}\bar{\phi}\partial\phi + \frac{1}{2}W'(\phi)\bar{W}'(\bar{\phi}) + \bar{\psi}M\psi. \tag{7.6}$$

For this model we use the superpotential

$$W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{3}g\phi^3. \tag{7.7}$$

The system has two bosonic ground states which lead to a nonzero Witten index [116], therefore supersymmetry is never spontaneously broken in the $\mathcal{N} = (2,2)$ Wess-Zumino model.

A characteristic feature of the $\mathcal{N} = 1$ Wess-Zumino model in four dimensions survives the dimensional reduction, namely that bosonic and fermionic loop corrections cancel in such a way that the effective superpotential receives no quantum corrections. This is called the non-renormalisation theorem [157, 158, 159]. In the two-dimensional model the cancellations even render the model finite and allow for a direct comparison between the results from the FRG and from lattice calculations.

7.2 Supersymmetric flow equations

As an ansatz for the effective action we use an expansion in superspace[2]

$$\begin{aligned}\Gamma_k &= -2\int d^2x \int dy\, d\bar{y}\; Z_k^2(\partial\bar{\partial})\bar{\Phi}\Phi - 2\int d^2x \left[\int dy\; W_k(\Phi) + \int d\bar{y}\; \bar{W}_k(\bar{\Phi})\right] \\ &= \int \frac{d^2p}{4\pi^2}\left[Z_k^2(p^2)\left(2p^2\bar{\phi}\phi + \bar{\psi}\mathrm{i}\not{p}\psi - \frac{1}{2}\bar{F}F\right) + \frac{1}{2}FW_k' + \frac{1}{2}\bar{F}\,\bar{W}_k' + \bar{\psi}\left(W_k''P_+ + \bar{W}_k''P_-\right)\psi\right]. \quad (7.8)\end{aligned}$$

In the following we will use the real and imaginary parts ϕ_1, ϕ_2, F_1, F_2 instead of the complex coordinates.

In contrast to the usual super-covariant derivative expansion used in the previous chapters, here we only included those combinations of the supercovariant derivatives that merely reduce to space-time derivatives. As we shall discuss in section 7.2.1 a momentum dependence in $W_k(\phi)$ is irrelevant. An arbitrary Kähler potential $K\left(\bar{\Phi},\Phi\right)$ integrated over the whole superspace is not taken into account here, since we expect only a small influence for the renormalised mass from this. Another contribution neglected in this truncation comes from the terms of higher than quadratic order in the auxiliary field and the corresponding supersymmetric partner terms. We denote them as auxiliary field potential.

For the scale-dependent effective action (7.8) the auxiliary fields obey the equations of motion $F = \bar{W}_k'(\phi)/Z_k^2$ and $\bar{F} = W_k'(\phi)/Z_k^2$. This leads to the on-shell action

$$\Gamma_k^{\text{on}} = \int \frac{d^2p}{4\pi^2}\left[\frac{1}{2}Z_k^2(p^2)p^2\phi\bar{\phi} + \frac{1}{2}\frac{|W_k'|^2}{Z_k^2(p^2)} + \mathrm{i}Z_k^2(p^2)\bar{\psi}\not{p}\psi + \bar{\psi}(W_k''P_+ + \bar{W}_k''P_-)\psi\right]. \quad (7.9)$$

Supersymmetry is preserved if the mass is shifted by a momentum-dependent infrared regulator[3], $m \to m + Z_k^2 \cdot r_1(k,p^2)$ or the wave-function renormalization is multiplied by a momentum-dependent regulator function, $Z_k^2 \to Z_k^2 \cdot r_2(k,p^2)$. Such regulators are the same as the ones used in the previous chapters. To obtain a regularised path integral, R_k is included in

[2] see appendix E.1 for conventions in superspace. For the Fourier transformation we use the convention $\partial_j \to \mathrm{i}p_j$ with the notations $\boldsymbol{p} = (p_1,p_2)^T$ and $p = |\boldsymbol{p}|$ where there is no risk of misunderstandings.

[3] The regulator function is multiplied with the wave-function renormalization to ensure reparameterisation invariance of the flow equation.

terms of the cutoff action ΔS_k. In matrix notation the cutoff action reads

$$\Delta S_k = \frac{1}{2} \int \frac{d^2 p}{4\pi^2} \, \bar{\Psi} \, Z_k^2 R_k^T \, \Psi^T \tag{7.10}$$

with $\Psi = (\phi_1 \; \phi_2 \; F_1 \; F_2 \; \psi(-p)^T \; \bar{\psi}(p))$ and

$$R_k = \begin{pmatrix} R_k^B & 0 \\ 0 & R_k^F \end{pmatrix} \text{ with } R_k^B = \begin{pmatrix} p^2 r_2 \cdot \mathbb{1} & r_1 \cdot \sigma_3 \\ r_1 \cdot \sigma_3 & -r_2 \cdot \mathbb{1} \end{pmatrix} \text{ and } R_k^F = \begin{pmatrix} 0 & \mathrm{i}\not{p} \cdot r_2 - r_1 \cdot \mathbb{1} \\ \mathrm{i}\not{p} \cdot r_2 + r_1 \cdot \mathbb{1} & 0 \end{pmatrix}. \tag{7.11}$$

Inserting ansatz (7.8) in the flow equation (6.13), the scale-dependent propagator can be calculated along the lines described in [74]: The fluctuation matrix $\Gamma_k^{(2)} + R_k$ is decomposed into the propagator $\Gamma_0^{(2)} + R_k$ including the regulator functions and a part $\Delta\Gamma_k$ containing all field dependencies. The flow equation (6.13) is expanded in the number of fields. See appendix E.2 for the expansion and the explicit matrices.

7.2.1 Flow equation for the superpotential – The non-renormalisation theorem

As in previous chapters the scale-dependent superpotential is obtained by a projection on the terms linear in the auxiliary fields. We can either choose the real or imaginary part of the auxiliary field as they are bound to give the same results due to supersymmetry.

The superpotential $W(\phi) = u(\phi_1, \phi_2) + \mathrm{i}v(\phi_1, \phi_2)$ is a holomorphic function of ϕ_1 and ϕ_2, and therefore its real and imaginary part obey the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial \phi_1} = \frac{\partial v}{\partial \phi_2}, \quad \frac{\partial u}{\partial \phi_2} = -\frac{\partial v}{\partial \phi_1}. \tag{7.12}$$

Using these equations all contributions to the flow equations of the superpotential cancel and the flow equation simply reads

$$\partial_k u_k = 0, \; \partial_k v_k = 0 \;\Rightarrow\; \partial_k W_k = \partial_k \bar{W}_k = 0, \tag{7.13}$$

such that the superpotential remains unchanged during the RG flow. The Kähler potential does therefore not influence the flow of the superpotential, as found in [158]. Even the nontrivial momentum dependence considered here does not change this result.

Terms up to F^3 can directly influence the flow equation. Here, we only consider a truncation up to terms quadratic in the auxiliary field such that the non-renormalisation theorem is not fully proven but only in the truncation considered. Contributions from higher-order operators have to vanish among themselves. This result is similar to the proofs in four dimensions discussed in [43] and [160].

As the flow vanishes at leading order, the first quantity with a non-vanishing flow is the wave-function renormalisation which is a term at next-to-leading order in the considered truncation. It will turn out later that the momentum dependence is important for the renormalised mass (cf. section 7.3) therefore we have already included it in the ansatz (7.8).

7.2.2 Momentum-dependent flow equation for the wave-function renormalisation

The flow equation for the wave-function renormalisation can be obtained from a projection onto the terms quadratic in the auxiliary fields. It is derived in appendix E.2 and reads

$$\begin{aligned}\partial_k Z_k^2(p) &= -8g^2 \int \frac{d^2q}{4\pi^2} \frac{h(\boldsymbol{p}-\boldsymbol{q})h(\boldsymbol{q})}{\nu(\boldsymbol{q})^2\nu(\boldsymbol{p}-\boldsymbol{q})^2} \left[\partial_k R_1(\boldsymbol{q}-\boldsymbol{p})M(\boldsymbol{p}-\boldsymbol{q})\nu(\boldsymbol{q}) + \partial_k R_1(\boldsymbol{q})M(\boldsymbol{q})\nu(\boldsymbol{p}-\boldsymbol{q})\right] \\ &+ 4g^2 \int \frac{d^2q}{4\pi^2} \frac{h(\boldsymbol{p}-\boldsymbol{q})\partial_k R_2(\boldsymbol{q})u(\boldsymbol{q})\nu(\boldsymbol{p}-\boldsymbol{q})}{\nu(\boldsymbol{q})^2\nu(\boldsymbol{p}-\boldsymbol{q})^2} + 4g^2 \int \frac{d^2q}{4\pi^2} \frac{h(\boldsymbol{q})\partial_k R_2(\boldsymbol{q}-\boldsymbol{p})\nu(\boldsymbol{q})u(\boldsymbol{p}-\boldsymbol{q})}{\nu(\boldsymbol{q})^2\nu(\boldsymbol{p}-\boldsymbol{q})^2}\end{aligned} \tag{7.14}$$

with the abbreviations (recall that $|\boldsymbol{q}| = q$)

$$\begin{aligned}h(\boldsymbol{q}) &= \left(r_2\left(k,q\right)+1\right) Z_k^2\left(q\right),\ M(\boldsymbol{q}) = m + r_1(k,q)Z_k^2\left(q\right),\ R_i(\boldsymbol{q}) = r_i\left(k,q\right) Z_k^2\left(q\right), \\ u(\boldsymbol{q}) &= M(\boldsymbol{q})^2 - q^2h^2(\boldsymbol{q}),\ \nu(\boldsymbol{q}) = M(\boldsymbol{q})^2 + q^2h^2(\boldsymbol{q}).\end{aligned} \tag{7.15}$$

The model is a ultraviolet-finite theory and therefore it is sufficient to use the simple, mass-like infrared regulator

$$r_1\left(k,q^2\right) = k \quad \text{and} \quad r_2\left(k,q^2\right) = 0. \tag{7.16}$$

After a shift in the integration variables in the second part of the integral (7.14) the flow equation simplifies to

$$\partial_k Z_k^2(p) = -16g^2 \int \frac{d^2q}{4\pi^2} \frac{kZ_k^2\left(q\right)+m}{N(\boldsymbol{q})^2 N(\boldsymbol{p}-\boldsymbol{q})} Z_k^2\left(q\right) Z_k^2\left(|\boldsymbol{p}-\boldsymbol{q}|\right) \partial_k \left(kZ_k^2\left(q\right)\right), \tag{7.17}$$

with the abbreviation $N(\boldsymbol{q}) = \left(q^2 Z_k^4\left(q\right) + (kZ_k^2\left(q\right)+m)^2\right)$. In order to deal with the partial differential equation we use a numerical toolbox called FlowPy. See [155] for details on the numerical setup.

In the next section we determine the renormalised masses from the non-perturbative wave-function renormalisation with full momentum dependence calculated with FlowPy.

Before turning to the actual calculation, we briefly discuss the errors that arise due to the numerical calculation of the wave-function renormalisation with FlowPy. For this we consider a one-loop perturbative calculation. It is possible to calculate the perturbative expression for

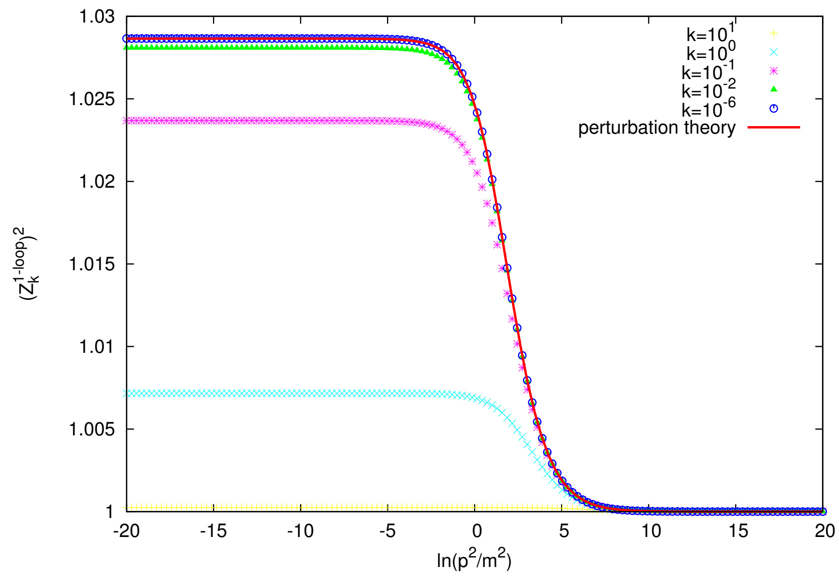


Figure 7.1: Perturbative flow for the parameters $\lambda = g/m = 0.3$ and $m = 1$. The solid line is the plot of equation (7.18).

$Z^2_{\text{one-loop}}(p)$ analytically from the perturbative flow equation by performing the k-integral using $\lim_{k\to\infty}(r_1, r_2) \to \infty$ and $\lim_{k\to 0}(r_1, r_2) \to 0$. This yields

$$Z^2_{\text{one-loop}} = 1 + \frac{g^2}{\pi^2}\int \frac{d^2q}{(m^2+q^2)(m^2+|q-p|^2)} = 1 + 4g^2\frac{\operatorname{artanh}\left(p(4m^2+p^2)^{-1/2}\right)}{\pi p\sqrt{4m^2+p^2}}\,. \tag{7.18}$$

In figure 7.1 the results of the perturbative flow calculated with FlowPy at different values of the RG-scale k and the analytic result (7.18) is shown. As can be seen, the numerical error due to discretisation and interpolation is very small. Therefore the wave-function renormalisation is considered to be exact in this truncation. We expect the errors caused by the truncation to be larger than the error introduced by the numerical calculations.

7.3 The renormalised mass

The analytic continuation of the bosonic propagator,

$$G_{\text{bos}}(p) = \frac{1}{p^2 + m^2 + \Sigma(p, m, g)}, \tag{7.19}$$

has a pole which defines the renormalised mass. Since the bare mass m is a parameter of the superpotential (7.7) it is not changed during the flow. Σ is the self-energy. As expected from a supersymmetric theory, the pole of the fermionic propagator leads to the same renormalised mass as the bosonic propagator.

The Fourier transformation of $G_{\text{bos}}(p)$ yields the correlator

$$C_{\text{bos}}(x_1) = \int \frac{dp_1}{2\pi} G_{\text{bos}}(p_1, 0) e^{ip_1 x_1}. \tag{7.20}$$

The renormalised mass can be obtained from the long-range exponential decay of this quantity and we denote it as *correlator* mass m_{corr} in the following. We can also define a renormalised mass, denoted as *propagator* mass, through $m_{\text{prop}}^2 = (G_{\text{bos}}(p))^{-1}|_{p=0}$. In the previous chapters, we used this definition because the wave-function renormalisation was independent of the momentum such that correlator and propagator mass were the same.

To compare the renormalised masses from the FRG with the results of the lattice simulation [151] we have to consider the masses of the particles in the on-shell theory. In the infrared limit the bosonic propagator from the on-shell action (7.9) reads in the present truncation

$$G_{\text{bos}}^{\text{NLO}}(p) = \frac{1}{p^2 Z_{k\to 0}^2(p^2) + m^2 / Z_{k\to 0}^2(p^2)}. \tag{7.21}$$

The fermionic propagator reads

$$G_{\text{ferm}}^{\text{NLO}}(p) = \frac{\not{p}}{p^2 Z_{k\to 0}^4(p^2) + m^2}. \tag{7.22}$$

Both propagators have the same poles and therefore lead, as expected, to the same renormalised masses for bosons and fermions.

For a small self-energy Σ a comparison between equation (7.19) and (7.21) leads to the approximate relation

$$Z_{k\to 0}^2(p) = 1 + \frac{\Sigma(p, m, g)}{p^2 - m^2}. \tag{7.23}$$

For the propagator mass the fields in the on-shell action have to be rescaled with the wave-function renormalisation such that the kinetic term is of the canonical form. Neglecting the momentum dependence in the wave-function renormalisation leads to

$$m_{\text{prop}} = \frac{m}{Z_{k\to 0}^2(p=0)}. \tag{7.24}$$

A numerical calculation can provide $Z_k^2(p)$ only for real p and its analytic continuation cannot be determined straightforwardly. Instead, the discrete Fourier transformation of $G_{\text{bos}}^{\text{NLO}}(p)$ with momenta $p = \left\{0, \frac{2\pi}{aN}, \ldots, \frac{2\pi(N-1)}{aN}\right\}$ on the interval $x \in [0, aN = L]$ is considered. For distances much smaller than L this should approximate $C_{\text{bos}}^{\text{NLO}}(x)$ in a well defined way. More precisely, instead of the exponential decay we obtain the long distance behaviour

$$C_{a,m_{\text{cor}}}(x_1) \propto \cosh\left(m_{\text{corr}}\left(x_1 - \frac{L}{2}\right)\right) \tag{7.25}$$

after the integration over the spatial direction. The mass can be determined from a fit to this function, as it is done in lattice simulations. The details of this procedure can be found in appendix E.3.

With the analytic result (7.18) for $Z^2_{\text{one-loop}}$ at hand the poles of $G^{\text{NLO}}_{\text{bos}}(p)$ can be calculated to obtain a perturbative approximation of m_{corr}. Note that this analytic solution of the perturbative flow together with equation (7.23) leads to the same result as a one-loop *on-shell* calculation of the polarisation Σ (cf.[133]). Expanding the pole of the propagator (7.19) to first order in the dimensionless parameter $\lambda^2 = g^2/m^2$ leads to the one-loop approximation of the renormalised mass

$$(m^{\text{one-loop}}_{\text{corr}})^2 = m^2 \left(1 - \frac{4}{\sqrt{27}}\lambda^2 + O\left(\lambda^4\right)\right) . \tag{7.26}$$

However, keep in mind that this expansion is only valid for small λ.

7.3.1 Weak couplings

Let us start with an investigation of the weak coupling sector which is defined as $\lambda < 0.3$, where perturbation theory provides an excellent cross-check to establish the correctness of the ansatz and the errors in the numerical approximation.

The bare mass in the lattice simulations [151] is taken to be $m = 15$. Note the following concerning the units of the mass: In the lattice calculation, the mass is measured in units of the box size, i. e. the physical volume of the lattice simulation. Similarly, everything can be reformulated in terms of the dimensionless ratio of bare and renormalised mass. For the numerical treatment of equation (7.17) dimensionless quantities have to be used. Because of the non-renormalisation theorem the bare quantities in the superpotential enter in the flow equation only as parameters. Rescaling the dimensionful quantities with the bare mass sets the scale in this model to $m = 1$. To get the same units as in the lattice simulations the resulting renormalised mass is multiplied by 15.

The correlator masses in the weak coupling regime are calculated with the momentum-dependent wave-function renormalisation from the flow equation (7.17) solved with FlowPy. The technical details of the determination of the correlator masses are described in appendix E.3. The results are shown in the second column of table 7.1. The values in the fourth column are taken from a Monte-Carlo simulation on the lattice [151]. Note that the lattice and perturbative results agree within the statistical errors. Hence perturbation theory already provides a good cross-check for the results from the flow equation.

In figure 7.2 the correlator masses from the flow equation, the lattice simulation and the one-loop result (7.26) for m_{corr} are shown. The masses calculated from the flow equation agree very well with perturbation theory and with the results from lattice simulations. This can be

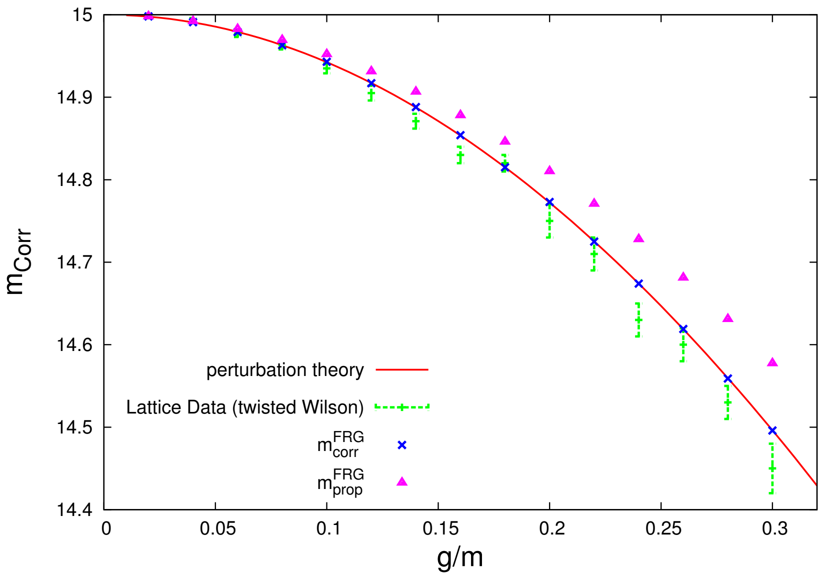


Figure 7.2: Comparison between lattice data taken from [151] and the results for the correlator mass $m_{\text{corr}}^{\text{FRG}}$ *with* momentum dependence and $m_{\text{prop}}^{\text{FRG}}$ without momentum dependence in the weak coupling regime.

λ	$m_{\text{corr}}^{\text{FRG}}$	$m_{\text{prop}}^{\text{FRG}}$	$m_{\text{corr}}^{\text{lattice}}$	λ	$m_{\text{corr}}^{\text{FRG}}$	$m_{\text{prop}}^{\text{FRG}}$	$m_{\text{corr}}^{\text{lattice}}$
0.02	14.998	14.998	14.999(1)	0.18	14.815	14.846	14.82(1)
0.04	14.991	14.992	14.993(3)	0.20	14.773	14.810	14.75(2)
0.06	14.979	14.983	14.977(4)	0.22	14.674	14.771	14.71(2)
0.08	14.963	14.970	14.963(5)	0.24	14.674	14.728	14.63(2)
0.10	14.943	14.952	14.935(6)	0.26	14.619	14.681	14.60(2)
0.12	14.917	14.931	14.905(9)	0.28	14.559	14.631	14.53(2)
0.14	14.888	14.907	14.871(9)	0.30	14.496	14.578	14.45(3)
0.16	14.854	14.878	14.83(1)				

Table 7.1: Renormalised masses obtained with the flow equation with and without momentum dependence ($m_{\text{corr}}^{\text{FRG}}$ and $m_{\text{prop}}^{\text{FRG}}$) as well as lattice data $m_{\text{corr}}^{\text{lattice}}$ from a continuum extrapolation [151] in the weak coupling regime.

quantified by comparing the correction to the bare mass $\Delta m_{\text{corr}} = m - m_{\text{corr}}$. This yields

$$\frac{\Delta m_{\text{corr}}^{\text{FRG}}}{\Delta m_{\text{corr}}^{\text{lattice}}} \simeq 0.95. \tag{7.27}$$

Taking into account the statistical error of the lattice data no significant difference to the FRG results can be found.

To conclude, in the weak coupling regime the truncation of the flow equation with full momentum dependence suffices to capture the main aspects of the model. Higher-order operators, which yield an auxiliary field effective potential, have little influence, as expected.

To investigate the influence of the momentum dependence in the wave-function renormalisation, the propagator mass (7.24) is calculated. The results are shown in the third column of table 7.1 and in figure 7.2. A comparison between the propagator mass and the correlator mass from the lattice calculation yields

$$\frac{\Delta m_{\text{prop}}^{\text{FRG}}}{\Delta m_{\text{corr}}^{\text{lattice}}} \simeq 0.75. \tag{7.28}$$

Already in the weak coupling regime it is necessary to include the momentum dependence in order to determine the corrections to the renormalised mass with satisfying accuracy.

7.3.2 Intermediate couplings

At intermediate couplings $0.3 \leq \lambda \leq 1$ perturbation theory is no longer reliable and we have to use lattice calculations for a comparison instead. For a discussion of difficulties that arise in the lattice formulation of this model see [15, 151, 161].

For intermediate couplings the nonlocal *SLAC* discretisation and the *Twisted Wilson* discretisation provides the most reliable results, cf. [151] for details. The renormalised masses of these discretisation are used for a comparison with the results from the FRG. They are shown in the third and fourth column of table[4] 7.2 and displayed in figure 7.3 (boxes with error bars) together with the order λ^2 expanded result (7.26) for m_{corr} (dashed line). Keep in mind, however, that perturbation theory is no longer reliable in this regime. The good agreement between the perturbation theory expanded to order λ^2 and the lattice results is a coincidence. In fact, the result at $O(\lambda^2)$ has to fail for large values of λ because otherwise the renormalised masses would become negative.

The correlator masses determined from the FRG are shown in the second column of table 7.2 and displayed in figure 7.3 (lying crosses). Additionally the perturbative result for the renormalised mass is shown (solid line). It is determined from the pole of the propagator (7.19) with the

[4] All lattice results are extrapolated to the continuum.
[5] C. Wozar, private communication

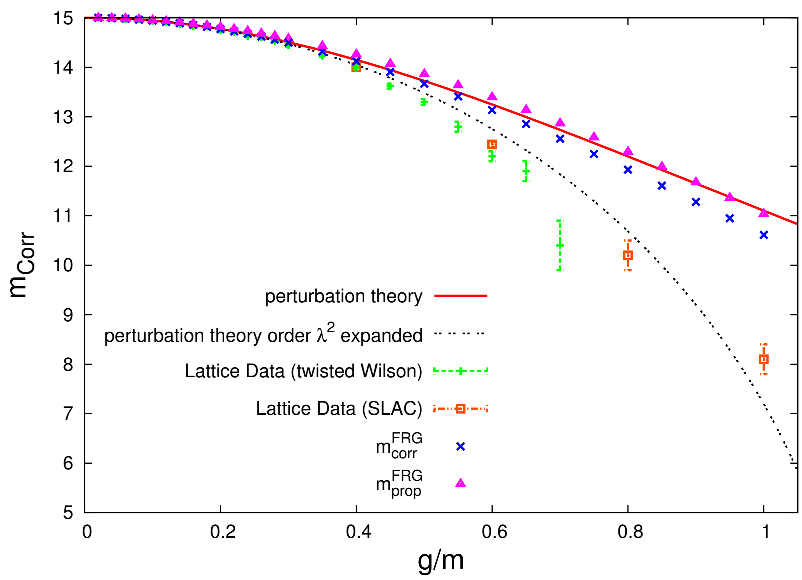


Figure 7.3: Comparison between lattice data taken from [151] and the results for the correlator mass $m_{\text{corr}}^{\text{FRG}}$ *with* momentum dependence and $m_{\text{prop}}^{\text{FRG}}$ without momentum dependence in the intermediate coupling regime

λ	$m_{\text{corr}}^{\text{FRG}}$	$m_{\text{prop}}^{\text{FRG}}$	tw. Wilson	SLAC imp.
0.35	14.321	14.428	14.23(2)	
0.40	14.123	14.259	13.99(3)	14.00(1)
0.45	13.905	14.069	13.62(5)	
0.50	13.666	13.861	13.30(6)	
0.55	13.411	13.636	12.8(1)	
0.60	13.138	13.394	12.2(1)	12.44(6)
0.65	12.854	13.137	11.9(2)	
0.70	12.556	12.866	10.4(5)	
0.75	12.248	12.583		
0.80	11.932	12.290		10.2(3)
0.85	11.609	11.987		
0.90	11.280	11.676		
0.95	10.948	11.358		
1.00	10.613	11.036		8.1(3)[5]

Table 7.2: Masses obtained with the flow equation with and without momentum dependence $m_{\text{corr}}^{\text{FRG}}$ and $m_{\text{prop}}^{\text{FRG}}$) as well as lattice data [151] in the regime with intermediate couplings.

perturbative one-loop result for the self-energy [155]:

$$\Sigma(p) = \frac{4g^2(m^2 - p^2)}{\pi p\sqrt{4m^2 + p^2}} \operatorname{artanh}\left(p(4m^2 + p^2)^{-1/2}\right) \tag{7.29}$$

Although the corrections to the bare mass from the wave-function renormalisation with full momentum dependence capture some of the quantum effects, they do not account for *all* the non-perturbative effects present in this model. To quantify this, these corrections are compared to the corrections found in lattice calculations. This yields results between

$$\frac{\Delta m_{\text{corr}}^{\text{FRG}}}{\Delta m_{\text{corr}}^{\text{lattice}}} \simeq 0.9 \quad \text{for} \quad \lambda = 0.35 \quad \text{and} \quad \frac{\Delta m_{\text{corr}}^{\text{FRG}}}{\Delta m_{\text{corr}}^{\text{lattice}}} \simeq 0.65 \quad \text{for} \quad \lambda = 1.0. \tag{7.30}$$

The fact that the wave-function renormalisation accounts for less of the quantum corrections as the coupling grows is due to the growing influence of higher-order operators, especially the auxiliary field potential. In the present truncation only terms that are at most quadratic in the auxiliary field are considered and back-reactions from a potential for the auxiliary field are neglected. As can be seen from a diagrammatic expansion of the flow equation, terms up to order F_i^4 directly modify the flow equation for the wave-function renormalisation[6], which is proportional to F_i^2. As already seen in the previous chapters the influence of higher-order operators grows with the strength of the couplings. A truncation that goes beyond the momentum-dependent wave-function renormalisation has to be considered to improve the results in the regime with intermediate couplings.

The results for the propagator mass are shown in the third column of table 7.2 and in figure 7.3 (triangles). The comparison to the lattice results yields

$$\frac{\Delta m_{\text{prop}}^{\text{FRG}}}{\Delta m_{\text{corr}}^{\text{lattice}}} \simeq 0.75 \quad \text{for} \quad \lambda = 0.35 \quad \text{and} \quad \frac{\Delta m_{\text{prop}}^{\text{FRG}}}{\Delta m_{\text{corr}}^{\text{lattice}}} \simeq 0.6 \quad \text{for} \quad \lambda = 1.0. \tag{7.31}$$

The improvement due to the momentum dependence in Z_k^2 is not as pronounced as it is in the weak coupling regime because other operators are more important in this regime than the momentum dependence.

7.4 Beyond next-to-leading order

The non-renormalisation theorem in the context of the FRG formulation emerges in a very simple form, namely that the flow equation for the superpotential vanishes identically. To prove this, all that is needed is that the superpotential is a holomorphic function. As a consequence, the

[6] The argumentation is the same as for the superpotential in appendix F. As the wave-function renormalisation has two external auxiliary field lines, it is not possible to construct a one-loop contribution from an operator with more than four auxiliary fields.

renormalisation of the mass and the other coupling constants is caused by the wave-function renormalisation and higher-order operators that are not restricted by the non-renormalisation theorem. In the weak coupling regime the wave-function renormalisation in the present truncation – with full momentum dependence – accounts for all the quantum effects calculated with the lattice.

For intermediate couplings this is no longer the case and only a part of the quantum corrections are actually captured. In this regime a truncation that includes higher-order operators in the auxiliary field has to be considered. This is not surprising because the influence of higher-order operators increases as the coupling grows as we have already seen in the previous chapters (cf. chapter 4).

Even though we have not done it here, the methods and results of this chapter can easily be applied to the $\mathcal{N} = 1$ Wess-Zumino model in four dimensions. Especially the non-renormalisation theorem can be derived in exactly the same way in four dimensions. However, the four dimensional theory is no longer ultraviolet finite, which means that it is not so easy to compare results with the ones from lattice calculations such as results presented by C. Chen, E. Dzienkowski and J. Giedt [162].

The FRG is able to disentangle contributions to the quantum corrections caused by higher-order operators from contributions due to the (momentum-dependent) wave-function renormalisation. However, an inclusion of these operators poses a great challenge as a lot of terms are involved. This ambitious goal is not pursued here further but left as an interesting question for future work.

8 Conclusions and outlook

This work aims at an extension of the functional renormalisation group framework to supersymmetric theories such that it can eventually be used to study supersymmetric extensions of the standard model.

In order to preserve supersymmetry we have formulated the flow equations, including the cutoff action, in superspace. The truncation of the action has been performed in superspace as well, implying that only functions of superfields and covariant derivatives enter in the ansatz for the truncated action. If we work in components, it follows from the formulation in superspace that we have to use the off-shell formulation of the theory which includes an auxiliary field.

The regulator structure that preserves supersymmetry deviates from the one usually used for non-supersymmetric theories with Yukawa interactions. In a supersymmetric theory the bosonic and the fermionic regulator are tightly connected in order to keep supersymmetry intact. Additionally also the auxiliary field has to be regularised which implies that it becomes a dynamic field. We can no longer use a simple derivative expansion in this setup since this would break supersymmetry. Instead, an expansion in super-covariant derivatives provides a systematic expansion scheme. The quantity at leading order in this expansion is the superpotential, the quantity occurring at next-to-leading order is the wave-function renormalisation. The flow of both quantities in a component formulation can be read off from a projection on the part linear or quadratic in the auxiliary field respectively.

In chapter 4 we have first investigated supersymmetric quantum mechanics. A lot of results on this model are known such that it is an ideal test candidate for a first application of the FRG to a supersymmetric theory. In this work the case where SuSy is always unbroken is considered. As a benchmark test the energy of the first excited state is calculated. Without wave-function renormalisation we find quite a large deviation from the exact results. The results are considerably improved by including a wave-function renormalisation. This is not surprising because the supercovariant derivative expansion mixes different orders of momentum. Through the auxiliary field the wave-function renormalisation modifies directly the flow equation for the superpotential.

Having established in which way the FRG can be extended to supersymmetric field theories we have investigated the $\mathcal{N} = 1$ Wess-Zumino model in two dimensions in chapter 5. The approach presented in the supersymmetric quantum mechanics can easily be generalised to this model. In this chapter we have concentrated on a superpotential that allows for spontaneous SuSy breaking. For this model the main focus was on the phase transition between the phase with broken and unbroken supersymmetry and on the fixed-point structure. It is known from bosonic theories that two-dimensional theories show a behaviour quite different from those in higher dimensions because the bosonic field is dimensionless. As a consequence, an investigation of the fixed-point structure of the supersymmetric model revealed that in the LPA only a continuum of periodic fixed points is accessible whereas in the NLO we have found a discrete set of solutions classified by the number of nodes. The fixed-point structure at NLO survives in three dimensions.

The model exhibits one fixed point with only one infrared-unstable direction. All trajectories are attracted to this fixed point. All other fixed points have an increasing number of infrared-unstable directions. The phase transition is driven by the one infrared-unstable direction. By fine-tuning the unstable direction we can reach the critical point corresponding to the phase transition. Special to this model is a connection between the critical exponent belonging to the infrared-unstable direction and the anomalous dimension, called superscaling relation. As a consequence of the superscaling relation the bosonic mass scales with the RG scale and it vanishes as the scale is lowered to the infrared.

We find that the phase diagram spanned by the bare coupling $(\bar{\lambda}_\Lambda, \bar{\lambda}_\Lambda \bar{a}_\Lambda^2)$ is divided in two distinct regions namely the one with broken and the one with unbroken supersymmetry. In accordance with a qualitative argument by Witten [116] we find that there exists a maximal value for the bare couplings $\bar{a}_\Lambda^2$ above which supersymmetry breaking is not possible.

We have also calculated the critical exponents and the behaviour of the scalar mass in the regime with broken SuSy. We have found that the scalar mass is proportional to the RG-scale and therefore vanishes as the RG-scale is lowered to the infrared. In this respect, our results go beyond the lattice results. Previously the phase transition value was calculated on the lattice for just a few values of λ whereas critical exponents have never been calculated for this model before. Our predictions for the scaling of the mass can be verified by lattice simulations. First results in this direction [16] are very encouraging and seem to confirm the existence of a massless bosonic phase.

Based on the results from the two-dimensional model we have investigated the three-dimensional $\mathcal{N} = 1$ Wess-Zumino model in chapter 6. The flow equations derived in two dimensions generalise to higher dimensions straightforwardly. Both models have a lot of similarities but also some differences. The most striking one is that in three dimensions we find only two fixed points: The Gaußian one with all couplings set equal to zero and the supersymmetric analogue of the Wilson-Fisher fixed point. In this model the SuSy phase transition is also driven by the infrared-unstable direction of the fixed point. Similar to both models is that even in three dimensions the superscaling relation holds. In these models this results in a mass which scales to zero as the RG scale is lowered to the infrared.

Compared to bosonic $O(N)$ models the most prominent difference is that the infrared-unstable direction does not influence the flow of the infrared-stable direction and therefore no fine-tuning is required to reach the fixed point for these stable directions. As a consequence, the theory in the broken phase is always massless in the infrared. The infrared-unstable direction plays the same role for the supersymmetry breaking as in two dimensions. Namely it has to be fine-tuned so that the system reaches the phase transition.

The three-dimensional model is investigated additionally at finite temperatures with the aid of the Matsubara formalism. The supersymmetry breaking due to different boundary conditions for fermions and bosons is manifest in the different flow equations for the bosonic and fermionic couplings. At finite temperatures it is possible to define a pressure which obeys the temperature dependence of the Stefan-Boltzmann law in three dimensions as expected from a theory with massless scalar fields.

Even though supersymmetry is explicitly broken at finite temperatures, the $\mathbb{Z}_2$ symmetry of the model can either be restored or broken at finite temperature. Whether $\mathbb{Z}_2$ symmetry is broken or not depends on the temperature and parameters of the model, i. e. the initial values of the couplings at the initial RG scale. Since supersymmetry and $\mathbb{Z}_2$ symmetry are intimately linked, a study of $\mathbb{Z}_2$ symmetry can be used to measure the strength of supersymmetry breaking. There exist two different phases at finite temperatures: One phase with soft supersymmetry breaking due to the different statistics of bosons and fermions but broken $\mathbb{Z}_2$ symmetry and one with restored $\mathbb{Z}_2$ symmetry.

We have discussed several similarities and differences of scalar $O(N)$ models and the $\mathcal{N} = 1$ Wess-Zumino model at zero and finite temperatures, e. g. the fixed-point structure at zero temperature and the behaviour at finite temperature. The phase diagram is very similar to the one in two dimensions, in particular we have found again a maximal value for a_Λ^2 above which SuSy breaking is not possible.

Chapter 7 deals with the two-dimensional $\mathcal{N} = (2,2)$ Wess-Zumino model. The model is finite and allows to directly compare the results to Monte Carlo simulations on the lattice.

In the local potential approximation the non-renormalisation theorem is found in a very simple form: The flow equation of the superpotential vanishes identically. The first quantity with a non-vanishing flow equation is the wave-function renormalisation. It causes a renormalisation of the mass. The renormalised mass has been calculated to high precision in lattice simulations [151]. In order to calculate the renormalised mass in the FRG with a satisfying accuracy we have had to include a nontrivial momentum dependence in the wave-function renormalisation even for small couplings. To solve the flow equation with full momentum dependence we have developed a numerical toolbox called FlowPy [155]. This allowed us to solve the differential equation with high numerical precision. With momentum dependence the agreement between lattice results and FRG calculations is very good in the weak coupling regime.

At intermediate couplings the wave-function renormalisation with full momentum dependence is not sufficient to capture all quantum effects calculated in the lattice simulations. They are generated by higher-order operators which are not restricted by the non-renormalisation theorem. In order to improve the agreement between lattice and FRG calculations we may need to include in the truncation a potential for the auxiliary fields and the supersymmetric partner terms, which are generated by higher-order operators. Taking at least terms to order F^3 into account should considerably improve the results since these terms modify the flow equation for the wave-function renormalisation directly.

It has become clear during our investigations that a potential for the auxiliary field plays an important role for the flow equations. An auxiliary field potential – with ϕ dependent couplings – is needed to make the bosonic potential, which is obtained after the auxiliary fields have been integrated out, convex. Such a potential is obtained from higher orders in the supercovariant derivative expansion. The investigation of the $\mathcal{N} = (2,2)$ Wess-Zumino model showed that higher orders in the auxiliary field are essential to find the correct values for the renormalised mass for intermediate couplings. In supersymmetric quantum mechanics we have found that the energy for the first excited state is not reproduced correctly as soon as the superpotential becomes non-convex if higher-order operators are neglected. An interesting challenge for future work is to implement such higher-order terms in the flow equations.

Currently, further investigations on other models based on this work are under way. We work for example on an extension of the functional renormalisation group to non-linear supersymmetric sigma models. The strategy is similar to the one by A. Codello and R. Percacci [163] for the bosonic non-linear sigma model. The problem is to find a supersymmetric background field expansion so that the flow equations can be calculated. The application of the FRG to supersymmetric sigma models is the topic of a diploma thesis by M. C. Mastaler [164]. To understand how additional superfields alter the properties of the flow equations first the attention is focused on linear supersymmetric sigma models.

In this work, we have demonstrated for scalar theories that the FRG can be extended in a way that keeps supersymmetry intact. Nevertheless, for a description of supersymmetric extensions of the standard model, gauge fields have to be treated in the supersymmetric FRG approach. To this end, we investigate $\mathcal{N} = 1$ super Yang-Mills theory in four dimensions. Some work in this direction has been done by S. Falkenberg and B. Geyer [39] who formulated the flow equations in a background field expansion in superspace, using a supersymmetric regulator. However, they only calculate the running coupling to one-loop order in perturbation theory.

With the investigation of the fixed-point structure and its relation to critical phenomena in supersymmetric scalar theories we hope that we have made a valuable contribution to the understanding of the phenomenon of supersymmetry breaking. Still, a lot of work in the understanding of supersymmetric theories remains to be done.

A The Clifford algebra

This section follows the conventions in the book by Bronstein [165] and an article by de Andrade and Toppan [166].

Let X be n-dimensional linear space over a field $\mathbb{K}$ and $B : X \times X \to \mathbb{K}$ a bilinear form on X. The inner product $u \vee w$ is defined as

$$u \vee w + w \vee u = 2B(u, w) \quad \forall u, w \in X \tag{A.1}$$

$$\alpha \vee u = u \vee \alpha = \alpha u \quad \forall \alpha \in \mathbb{K} \quad u \in X. \tag{A.2}$$

The *Clifford algebra* $\mathfrak{C}(X)$ over $\mathbb{K}$ with respect to the bilinear form $B(\cdot, \cdot)$ with multiplication $\vee$ satisfies the conditions

1. $\mathfrak{C}(X)$ contains $\mathbb{K}$ and X
2. Let $b_1, ..., b_n$ be a basis of X, then the ordered products $\{1, b_1, ..., b_n, b_{i_1} \vee b_{i_2} \vee \ldots \vee b_{i_r}\}$, $r = 2, ..., n$ form a basis of $\mathfrak{C}$ for $i_1 < i_2 < ... < i_r$ and $i_k = 1, ...n \quad \forall k$.

In the main part the Clifford algebra of the Minkowski and the Euclidean space is needed. The bilinear form is the metric and a representation of the Clifford algebra is given by the γ matrices. In order to define Lagrange functions and charge conjugation for spinors three unitary matrices $\mathcal{A}, \mathcal{B}, C$ with $\mathcal{A} = \prod_{i=1}^{t} \gamma^i$ and $C = \mathcal{B}^T \mathcal{A}$ are defined. t is the number of time directions (positive sign in the metric). The matrices have the following properties:

$$\mathcal{A}\gamma^\mu \mathcal{A}^\dagger = (-)^{t+1} (\gamma^\mu)^\dagger \tag{A.3}$$

$$\mathcal{B}\gamma^\mu \mathcal{B}^\dagger = \eta (\gamma^\mu)^* \tag{A.4}$$

$$C\gamma^\mu C^\dagger = \eta (-)^{t+1} (\gamma^\mu)^T \tag{A.5}$$

with $\eta = \pm 1$. $\mathcal{B}$ has the property $\mathcal{B}^T = \varepsilon \mathcal{B}$ with $\varepsilon \pm 1$ and $\mathcal{B}\mathcal{B}^* = \varepsilon \cdot \mathbb{1}$ respectively. Further

$$C^T = \varepsilon \eta^t (-)^{\frac{t}{2}(t-1)} C. \tag{A.6}$$

holds

The *Dirac conjugation* in flat space-time is defined as $\overline{\psi} = \psi^\dagger \mathcal{A}$. Together with $\mathcal{B}$ the *charge conjugation* is defined as $\psi^c = \mathcal{B}^\dagger \psi^*$.

A *Majorana spinor* χ satisfies the condition $\chi^c = \chi$. From $\chi^* = \mathcal{B}^T\chi = \varepsilon\mathcal{B}\chi$ it follows that the Majorana condition can only be fulfilled for $\varepsilon = 1$.

The Dirac conjugation of a charge conjugated spinor can be written as $\overline{\psi}^c = (\mathcal{B}^\dagger\psi^*)^\dagger\mathcal{A} = \varepsilon\psi^T C$. For a Majorana spinor this implies $\overline{\chi} = \chi^T C$. For the charge conjugation matrix itself it follows from equation (A.6) $\psi^c = \mathcal{B}^\dagger\psi^* = C^*\overline{\psi}^T = \varepsilon\eta^t(-)^{\frac{t}{2}(t-1)}C^\dagger\overline{\psi}^T$. From this the condition

$$\overline{\psi}^c\gamma^{(n)}\chi^c = \eta^t(-)^{\frac{t}{2}(t-1}\psi^T C\gamma^{(n)}C^\dagger\overline{\chi}^T \tag{A.7}$$

arises. This leads to the following symmetry relations for Majorana spinors:

$$\begin{aligned}
\overline{\psi}\chi &= \eta^t(-)^{\frac{t}{2}(t-1)}\psi^T\overline{\chi}^T = -\eta^t(-)^{\frac{t}{2}(t-1)}\overline{\chi}\psi \\
\overline{\psi}\gamma^\mu\chi &= -\eta^{(t+1)}(-)^{\frac{t}{2}(t+1)}\psi^T(\gamma^\mu)^T\overline{\chi}^T = \eta^{(t+1)}(-)^{\frac{t}{2}(t+1)}\overline{\chi}\gamma^\mu\psi \\
\overline{\psi}\gamma^{\mu\nu}\chi &= -\eta^t(-)^{\frac{t}{2}(t-1)}\psi^T(\gamma^{\mu\nu})^T\overline{\chi}^T = \eta^t(-)^{\frac{t}{2}(t-1)}\overline{\chi}\gamma^{\mu\nu}\psi
\end{aligned} \tag{A.8}$$

In two-dimensional Euclidean space time the Majorana representation is given by

$$\gamma_1 = \mathrm{i}\sigma_1, \quad \gamma_2 = \mathrm{i}\sigma_3. \tag{A.9}$$

The charge conjugation matrix for $\eta = -1$ and $\varepsilon = 1$ reads $C = \mathbb{1}$.

B Technical details for SuSy-QM

B.1 Inversion of the propagator

In this appendix we calculate the inverse propagator

$$G_k = G_{0,k} - G_{0,k}(\bar\psi M_1 + M_2\psi)G_{0,k} + G_{0,k}\left(M_1 G_{0,k} M_2 - M_2 G_{0,k} M_1 - M_3\right) G_{0,k}\bar\psi\psi\,. \tag{B.1}$$

To keep the expressions simple we use the block notation

$$A = \begin{pmatrix} A_{BB} & A_{BF} \\ A_{FB} & A_{FF} \end{pmatrix}. \tag{B.2}$$

The non-vanishing blocks that are needed for the inverse propagator have the form

$$\begin{aligned}
(G_{0,k}^{-1})_{BB} &= \begin{pmatrix} (1+r_2)q^2 + iFW_k^{(3)} & \mathrm{i}(W_k'' + r_1) \\ \mathrm{i}(W_k'' + r_1) & 1 + r_2 \end{pmatrix}, \\
(G_{0,k}^{-1})_{FF} &= \begin{pmatrix} 0 & (1+r_2)q + \mathrm{i}(W_k'' + r_1) \\ (1+r_2)q - \mathrm{i}(W_k'' + r_1) & 0 \end{pmatrix}, \\
M_{1FB} = -M_{1BF} &= \begin{pmatrix} \mathrm{i}W_k^{(3)} & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{2BF} = -M_{2FB}^T = \begin{pmatrix} 0 & \mathrm{i}W_k^{(3)} \\ 0 & 0 \end{pmatrix}, \quad M_{3BB} = \begin{pmatrix} -\mathrm{i}W_k^{(4)} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{B.3}$$

To calculate the full propagator G_k we must first calcualte the inverse of $G_{0,k}^{-1}$. It is block diagonal and the inverse reads for constant fields

$$(G_{0,k})_{BB} = \frac{1}{\Delta_B}\begin{pmatrix} (1+r_2) & -\mathrm{i}(W_k'' + r_1) \\ -\mathrm{i}(W_k'' + r_1) & (1+r_2)q^2 + \mathrm{i}FW_k^{(3)} \end{pmatrix} \quad \text{and} \quad (G_{0,k})_{FF} = \frac{1}{\Delta_F}(G_{0,k}^{-1})_{FF} \tag{B.4}$$

with the factors

$$\Delta_\mathrm{F} = (1+r_2)^2 q^2 + (W_k'' + r_1)^2 \quad \text{and} \quad \Delta_\mathrm{B} = \Delta_\mathrm{F} + \mathrm{i}(1+r_2)FW_k^{(3)}\,. \tag{B.5}$$

B.2 Flow equations from the bosonic and fermionic part

Since the regulator R_k is block-diagonal only the diagonal blocks of the dressed propagator enter the flow equation (4.19). These blocks can be calculated with the help of (B.1). Inserting the regulator yields

$$\mathrm{Str}\,(G_k\,\partial_k R_k) = \int \mathrm{d}\tau\,\left(H_0(\phi,F) + H_1(\phi,F)\bar\psi\psi\right) \tag{B.6}$$

with the functions

$$H_0(\phi,F) = -\mathrm{i}FW_k^{(3)}\int\frac{\mathrm{d}q}{2\pi}\,\frac{\partial_k r_2((1+r_2)^2q^2-(W_k''+r_1)^2)+2(1+r_2)\partial_k r_1(W_k''+r_1)}{\Delta_B\Delta_F} \tag{B.7}$$

$$\begin{aligned}H_1(\phi,F) = i\int\frac{\mathrm{d}q}{2\pi}&\left(\Delta_F W_k^{(4)} - 2(W_k^{(3)})^2(W_k''+r_1)\right)\times\\ &\times\frac{\partial_k r_2((1+r_2)^2q^2-(W_k''+r_1)^2)+2(1+r_2)\partial_k r_1(W_k''+r_1)}{\Delta_B^2\Delta_F}\\ +\mathrm{i}\int\frac{\mathrm{d}q}{2\pi}&(1+r_2)(W_k^{(3)})^2\,\frac{\partial_k r_1((1+r_2)^2q^2-(W_k''+r_1)^2)-2(1+r_2)q^2\partial_k r_2(W_k''+r_1)}{\Delta_B\Delta_F^2}\,.\end{aligned} \tag{B.8}$$

To project onto the flow for the superpotential, the flow equation is differentiated with respect to F and afterwards $F=\psi=\bar\psi=0$ is set. This yields

$$\partial_k W_k' = -\frac{\mathrm{i}}{2}\frac{\delta\Gamma_k}{\delta F}\bigg|_{F=0} = -\frac{W_k^{(3)}}{2}\int\frac{\mathrm{d}q}{2\pi}\,\frac{\partial_k r_2((1+r_2)^2q^2-(W_k''+r_1)^2)+2(1+r_2)\partial_k r_1(W_k''+r_1)}{\Delta_B^2}\,. \tag{B.9}$$

Alternatively the flow equation can be obtained by projecting on the coefficient of $\bar\psi\psi$. This way we obtain

$$\partial_k W_k'' = \frac{1}{2}H_1(F,\phi)\bigg|_{F=0}\,. \tag{B.10}$$

The two projection formulas (B.9) and (B.10) indeed give rise to identical flows, since

$$\frac{\delta^2 H_0(\phi,F)}{\delta\phi\delta F}\bigg|_{F=0} = \mathrm{i}H_1(\phi,F)\big|_{F=0}\,. \tag{B.11}$$

C Flow equations in Minkowski space

In this section we derive the Wetterich equation in Minkowski space. For the sake of simplicity only a real scalar field is considered in this appendix. The generalisation to other fields, such as fermion or gauge fields, is straightforward. The generating functional in Minkowski-space is given by

$$Z[J] = \int \mathcal{D}\varphi \; \mathrm{e}^{\mathrm{i}(S[\varphi]+(J,\varphi))} \,, \tag{C.1}$$

where J denotes the external source and $(J,\varphi) \equiv \int d^d x\, J(x)\varphi(x)$. The generating functional W for the connected two-point functions, the so-called Schwinger functional, reads[1]

$$W[J] = \mathrm{i} \ln Z[j] \,. \tag{C.2}$$

From this we obtain

$$\frac{\delta}{\delta J} W[J] = \mathrm{i}\frac{\delta}{\delta J} \ln Z[J] = -\frac{\int \mathcal{D}\phi \; \mathrm{e}^{\mathrm{i}(S+\int J\phi)}\, \phi}{\int \mathcal{D}\phi \; \mathrm{e}^{\mathrm{i}(S+(J,\varphi))}} = -\phi = -\langle \varphi \rangle. \tag{C.3}$$

The effective action is the Legendre transform of the Schwinger functional,

$$\Gamma[\phi] = -W[J] - (J,\phi) \,, \tag{C.4}$$

where ϕ is the classical field. Using $\frac{\delta}{\delta J} W[J] = -\phi$ we obtain the equation of motion for the field ϕ:

$$\frac{\delta \Gamma[\phi]}{\delta \phi} = -\int \mathrm{d}^d y \frac{\delta W[J]}{\delta J(y)} \frac{\delta J(y)}{\delta \phi(x)} - \int \mathrm{d}^d y \frac{\delta J(y)}{\delta \phi(x)} \phi(y) - J(x) = -J(x) \,. \tag{C.5}$$

The scale-dependent generating functional is defined as

$$Z_k[J] = \mathrm{e}^{-\mathrm{i}W_k[J]} = \mathrm{e}^{\mathrm{i}\Delta S_k[\frac{\delta}{\delta J}]}\, Z[J] = \int \mathcal{D}\varphi \; \mathrm{e}^{\mathrm{i}(S[\varphi]+\int_x \varphi J + \Delta S_k[\varphi])} \tag{C.6}$$

with

$$\Delta S_k[\phi] = \frac{1}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \varphi(-q) R_k(q) \varphi(q) \,. \tag{C.7}$$

Next, the scale dependent effective action is defined as

$$\Gamma_k[\phi] = -W_k[J] - \int \mathrm{d}^4 x\, J\phi - \Delta S_k[\phi] \,. \tag{C.8}$$

[1]The generating functional W should not be confused with the superpotential.

In order to properly formulate the flow equations in Minkowski space we have to take $k^2 = p_\mu p^\mu$ as flow parameter. Therefore the derivative with respect to 'RG time' $t = \ln\left({}^{k^2}/_{\Lambda^2}\right)$ is defined to be $\partial_t = 2k^2\partial_{k^2}$. Taking the derivative of $\Gamma_k[\phi]$ with respect to t yields

$$\partial_t\Gamma_k[\phi] = -\partial_t W_k[J] - \partial_t \int \mathrm{d}^4 x J\phi - \partial_t \Delta S_k[\phi]\,, \tag{C.9}$$

where we use that the source is independent of k. The derivative of W_k can be written as

$$\partial_t W_k[J] = \mathrm{i}\partial_t \ln Z[J] = \frac{1}{2Z[J]}\int \frac{\mathrm{d}^d q}{(2\pi)^d}(\partial_t R_k)\frac{\delta^2 Z[J]}{\delta J \delta J}\,.$$

Using the definition of the Schwinger functional, $Z[J] = \mathrm{e}^{-\mathrm{i}W_k}$, yields

$$\partial_t W_k[J] = \mathrm{e}^{\mathrm{i}W_k}\,\frac{1}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}(\partial_t R_k)\frac{\delta^2\,\mathrm{e}^{-\mathrm{i}W_k}}{\delta J \delta J}\,. \tag{C.10}$$

Now the integrand is rewritten by making use of

$$\frac{\delta^2\,\mathrm{e}^{-\mathrm{i}W_k}}{\delta J \delta J} = \frac{\delta}{\delta J}\,\mathrm{e}^{-\mathrm{i}W_k}(-i)\frac{\delta W_k}{\delta J} = \mathrm{e}^{-\mathrm{i}W_k}(-i)\frac{\delta W_k}{\delta J}(-\mathrm{i})\frac{\delta W_k}{\delta J} + \mathrm{e}^{-\mathrm{i}W_k}(-i)\frac{\delta^2 W_k}{\delta J\delta J}\,, \tag{C.11}$$

then equation (C.10) can be rewritten as follows:

$$\partial_t W_k[J] = \frac{1}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\partial_t R_k\Bigg(-\underbrace{\frac{\delta W_k}{\delta J}}_{-\phi}\underbrace{\frac{\delta W_k}{\delta J}}_{-\phi} - \mathrm{i}\frac{\delta^2 W_k}{\delta J \delta J}\Bigg) = -\Delta S_k - \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}R_k\frac{\delta^2 W_k}{\delta J\delta J}\,. \tag{C.12}$$

With this relation the variation of the effective action equation (C.9) takes the form

$$\partial_t\Gamma_k[\phi] = \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}(\partial_t R_k)\frac{\delta^2 W_k}{\delta J\delta J}\,. \tag{C.13}$$

We can rewrite $\frac{\delta^2 W_k[J]}{\delta J \delta J}$ in terms of the effective action:

$$\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} = -\frac{\delta J}{\delta \phi} - R_k \Rightarrow \frac{\delta J}{\delta\phi} = -\left(\frac{\delta^2 W_k}{\delta J\delta J}\right)^{-1} = -\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)\,. \tag{C.14}$$

Making use of

$$\delta(q-q') = \frac{\delta\phi(q)}{\delta\phi(q')} = -\frac{\delta}{\delta\phi}\frac{\delta W_k[J]}{\delta J} = -\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\delta^2 W_k[J]}{\delta J\delta J}\frac{\delta J}{\delta\phi}\,, \tag{C.15}$$

we obtain the Wetterich equation in Minkowski space:

$$\partial_t\Gamma_k[\phi] = \frac{\mathrm{i}}{2}\,\mathrm{Tr}\left[\partial_t R_k\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1}\right]\,. \tag{C.16}$$

D Flow equations at finite temperature

In order to preserve supersymmetry in the RG flow for vanishing temperature we must choose a regulator function which regularises the theory in the time-like and the space-like directions in the same way. In order to make apparent how soft SUSY-breaking due to finite temperature emerges, we use the same regulator for the finite-temperature and zero-temperature studies, i. e.

$$r_2 = \left(\frac{k}{|p|} - 1\right) \theta\left(\frac{p^2}{k^2} - 1\right), \quad r_1 = 0\,. \tag{D.1}$$

In the LPA, we obtain the finite-temperature flow equations straightforwardly from the zero-temperature flow equations by replacing p_0 by the Matsubara modes ν_n and ω_n of fermionic and bosonic fields respectively and replacing the integration over p_0 by a summation over the Matsubara modes. The contribution of the bosons to the RG flow then reads:

$$\partial_k W_k' = -\frac{1}{2} W_k''' \cdot T \sum_{n=-\infty}^{\infty} \int \frac{d^2 p_s}{4\pi^2} \frac{(k^2 - W_k''^2)\theta(k^2 - p_s^2 - \omega_n^2)}{[k^2 + W_k''^2]^2 \sqrt{p_s^2 + \omega_n^2}}\,, \tag{D.2}$$

where p_s denotes the momenta in space-like directions. Along the lines of e. g. [167] we use Poisson's sum formula,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} dq\, f(q) \exp(-2\pi i \ell q)\,, \tag{D.3}$$

in order to obtain

$$\partial_k W_k' = -\frac{1}{2} W_k''' \cdot T \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} dq \int \frac{d^2 p_s}{4\pi^2} \frac{(k^2 - W_k''^2)\theta(k^2 - p_s^2 - (2\pi q T)^2)}{[k^2 + W_k''^2]^2 \sqrt{p_s^2 + (2\pi q T)^2}} e^{-2\pi i q \ell}\,. \tag{D.4}$$

To computate of the three-dimensional integral, we substitute $q' = 2\pi T q$ and introduce spherical coordinates $p_s^1 = r \cos\vartheta \sin\varphi$, $p_s^2 = r \sin\vartheta \sin\varphi$, $q' = r \cos\vartheta$. The angular integrations yields

$$\partial_k W_k' = -\frac{1}{2} W_k''' T \sum_{\ell=-\infty}^{\infty} \int_0^k \frac{dr}{4\pi^2 T} \frac{k^2 - W_k''^2}{[k^2 + W_k''^2]^2} \frac{2T \sin\left(\ell r / T\right)}{\ell}\,. \tag{D.5}$$

Finally the integration over r leads to

$$\partial_k W_k' = -\frac{(k^2 - W_k''^2) W_k'''}{8\pi^2 (k^2 + W_k''^2)^2} 2T^2 \sum_{\ell=-\infty}^{\infty} \frac{1 - \cos\left(\ell k / T\right)}{\ell^2}\,. \tag{D.6}$$

Since we made use of Poisson's resummation formula to rewrite the sum over the thermal modes, the flow equation can be split into a zero-temperature and a finite-temperature contribution:

$$\partial_k W_k' = \frac{(W_k''^2 - k^2) W_k'''}{8\pi^2 (k^2 + W_k''^2)^2} \left(k^2 + 4T^2 \sum_{\ell=1}^{\infty} \frac{1 - \cos\left(k\ell/T\right)}{\ell^2} \right) = \frac{(W_k''^2 - k^2) W_k'''}{8\pi^2 (k^2 + W_k''^2)^2} \left(k^2 + g_{\text{bos}}(T) \right). \tag{D.7}$$

The contribution of the fermions to the RG flow of the model can be obtained along the lines of the derivation of the bosonic contribution and reads:

$$\partial_k W_k' = \frac{(W_k''^2 - k^2) W_k'''}{8\pi^2 (k^2 + W_k''^2)^2} \left(k^2 + 4T^2 \sum_{\ell=1}^{\infty} (-)^{\ell} \frac{1 - \cos\left(k\ell/T\right)}{\ell^2} \right) = \frac{(W_k''^2 - k^2) W_k'''}{8\pi^2 (k^2 + W_k''^2)^2} \left(k^2 + g_{\text{ferm}}(T) \right). \tag{D.8}$$

Introducing the dimensionless temperature $\tilde{T} = T/k$, we can rewrite the functions $g_{\text{bos}}(T)$ and $g_{\text{ferm}}(T)$ in terms of polylogarithms:

$$g_{\text{bos}}\left(\tilde{T}\right) = \frac{2}{3} k^2 \tilde{T}^2 \left[\pi^2 - 3\text{Li}_2\left(\text{e}^{-\text{i}/\tilde{T}}\right) - 3\text{Li}_2\left(\text{e}^{-\text{i}/\tilde{T}}\right) \right], \tag{D.9}$$

$$g_{\text{ferm}}(\tilde{T}) = -\frac{2}{6} k^2 \tilde{T}^2 \left[\pi^2 + 6\text{Li}_2\left(-\text{e}^{-\text{i}/\tilde{T}}\right) + 6\text{Li}_2\left(-\text{e}^{-\text{i}/\tilde{T}}\right) \right]. \tag{D.10}$$

Using the identity [168]

$$\text{Li}_2(-z) + \text{Li}_2\left(-\frac{1}{z}\right) = 2\text{Li}_2(-1) - \frac{1}{2}\ln^2(z) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(z), \tag{D.11}$$

the function $g_{\text{bos}}(T)$ simplifies further to

$$g_{\text{bos}}(\tilde{T}) = \tilde{T}^2 \left[\pi^2 + \ln^2\left(-\exp\left(\text{i}/\tilde{T}\right)\right) \right] = \pi T \left(\pi T - (2s_B + 1)^2 \pi T + (2s_B + 1) 2k \right) - k^2, \tag{D.12}$$

where we used that

$$\ln\left(\exp\left(\text{i}/\tilde{T} + \text{i}\pi\right)\right) = \frac{\text{i}}{\tilde{T}} - \text{i}\pi\,(2s_B + 1), \quad s_B \le \frac{1}{2\pi\tilde{T}} \le s_B + 1 \;\Rightarrow\; s_B = \left\lfloor \frac{k}{2\pi T} \right\rfloor. \tag{D.13}$$

Similarly, exploiting the relation

$$\ln\left(\exp\left(\text{i}/\tilde{T}\right)\right) = \frac{\text{i}}{\tilde{T}} - 2\text{i}\pi\, s_F, \quad s_F - \frac{1}{2} \le \frac{1}{2\pi\tilde{T}} \le s_F + \frac{1}{2} \;\Rightarrow\; s_F = \left\lfloor \frac{k}{2\pi T} + \frac{1}{2} \right\rfloor, \tag{D.14}$$

leads to the result

$$g_{\text{ferm}}(T) = -k^2 \left(1 - s_F \frac{2\pi T}{k} \right)^2 \tag{D.15}$$

for the fermions. As expected, the functions $g_{\text{bos}}(T)$ and $g_{\text{ferm}}(T)$ exhibit the same behavior as the threshold functions discussed in Ref. [146].

E Technical details for the $\mathcal{N} = (2,2)$ Wess-Zumino model

E.1 Two dimensional Euclidean $\mathcal{N} = (2,2)$ superspace

A detailed discussion of the underlying supersymmetry algebra and a construction of the superspace can be found e. g. in [93]. The superspace formulation is constructed out of the supersymmetry transformations. The transformations are

$$\begin{aligned}
&\delta\phi = \bar\psi_1\varepsilon_1 + \bar\varepsilon_1\psi_1,\ \delta\bar\psi_1 = -\frac{1}{2}F\bar\varepsilon_1 - \partial\phi\bar\varepsilon_2,\ \delta\bar\psi_2 = -\bar\partial\bar\phi\bar\varepsilon_1 - \frac{1}{2}\bar F\bar\varepsilon_2, \delta F = 2(\partial\bar\psi_1\varepsilon_2 - \bar\varepsilon_2\bar\partial\psi_1),\\
&\delta\bar\phi = \bar\psi_2\varepsilon_2 + \bar\varepsilon_2\psi_2,\ \delta\psi_1 = -\frac{1}{2}F\varepsilon_1 + \bar\partial\phi\varepsilon_2,\ \delta\psi_2 = \partial\bar\phi\varepsilon_1 - \frac{1}{2}\bar F\varepsilon_2,\ \ \delta\bar F = 2(\partial\bar\psi_2\varepsilon_1 - \bar\varepsilon_1\bar\partial\psi_2).
\end{aligned} \tag{E.1}$$

We construct the superfield from its lowest component $\phi = \Phi(z,\bar z,0,0)$ by acting with the exponential function on this component [169]:

$$\Phi = \exp(-\delta_\varepsilon)\phi = \sum_{n=0}^{4}\frac{1}{n!}(-\delta_\varepsilon)^n\phi \tag{E.2}$$

This implies for the chiral superfield:

$$\begin{aligned}
\delta_\varepsilon\phi =&\bar\psi_1\varepsilon_1 + \bar\varepsilon_1\psi_1\\
\delta_\varepsilon^2\phi =&\delta\bar\psi_1\varepsilon_1 + \bar\varepsilon_1\delta\psi_1 = -F\bar\varepsilon_1\varepsilon_1 - \partial\phi\bar\varepsilon_2\varepsilon_1 + \bar\partial\phi\bar\varepsilon_1\varepsilon_2\\
\delta_\varepsilon^3\phi =&-\delta F\bar\varepsilon_1\varepsilon_1 - \partial\delta\phi\bar\varepsilon_2\varepsilon_1 + \bar\partial\delta\phi\bar\varepsilon_1\varepsilon_2 = -3\bar\partial\bar\psi_1\varepsilon_2\bar\varepsilon_1\varepsilon_1 + 3\partial\psi_1\bar\varepsilon_1\bar\varepsilon_2\varepsilon_1\\
\delta_\varepsilon^4\phi =&-3\bar\partial\delta\bar\psi_1\varepsilon_2\bar\varepsilon_1\varepsilon_1 + 3\partial\delta\psi_1\bar\varepsilon_1\bar\varepsilon_2\varepsilon_1 = 6\bar\partial\partial\phi\bar\varepsilon_2\varepsilon_2\bar\varepsilon_1\varepsilon_1
\end{aligned} \tag{E.3}$$

and the chiral superfield reads

$$\begin{aligned}
\Phi =&\phi - (\bar\psi_1\alpha_1 + \bar\alpha_1\psi_1) + \frac{1}{2}(-F\bar\alpha_1\alpha_1 - \partial\phi\bar\alpha_2\alpha_1 + \bar\partial\phi\bar\alpha_1\alpha_2) - \frac{1}{3!}(-3\bar\partial\bar\psi_1\alpha_2\bar\alpha_1\alpha_1 + 3\partial\psi_1\bar\alpha_1\bar\alpha_2\alpha_1)\\
&+ \frac{1}{4!}(6\bar\partial\partial\phi\bar\alpha_2\alpha_2\bar\alpha_1\alpha_1)\\
=&\phi - \bar\psi_1\alpha_1 - \bar\alpha_1\psi_1 - \frac{1}{2}F\bar\alpha_1\alpha_1 - \frac{1}{2}\partial\phi\bar\alpha_2\alpha_1 + \frac{1}{2}\bar\partial\phi\bar\alpha_1\alpha_2 + \frac{1}{2}\bar\partial\bar\psi_1\alpha_2\bar\alpha_1\alpha_1 - \frac{1}{2}\partial\psi_1\bar\alpha_1\bar\alpha_2\alpha_1\\
&+ \frac{1}{4}\bar\partial\partial\phi\bar\alpha_2\alpha_2\bar\alpha_1\alpha_1
\end{aligned} \tag{E.4}$$

The supercharges are

$$Q_1 = -\frac{\partial}{\partial\bar{\alpha}_1} + \frac{1}{2}\alpha_2\bar{\partial}, \quad \bar{Q}_1 = \frac{\partial}{\partial\alpha_1} - \frac{1}{2}\bar{\alpha}_2\partial, \quad Q_2 = -\frac{\partial}{\partial\bar{\alpha}_2} + \frac{1}{2}\alpha_1\partial, \quad \bar{Q}_2 = \frac{\partial}{\partial\alpha_2} - \frac{1}{2}\bar{\alpha}_1\bar{\partial}, \tag{E.5}$$

which reproduce the supersymmetry transformations:

$$\begin{aligned}
\delta\Phi =&(\bar{\varepsilon}Q + \bar{Q}\varepsilon)\Phi = (\bar{\varepsilon}_1 Q_1 + \bar{Q}_1\varepsilon_1 + \bar{\varepsilon}_2 Q_2 + \bar{Q}_2\varepsilon_2)\Phi \\
=&\bar{\varepsilon}_1\left(\psi_1 + \frac{1}{2}F\alpha_1 - \frac{1}{2}\bar{\partial}\phi\alpha_2 - \frac{1}{2}\bar{\partial}\bar{\psi}_1\alpha_2\alpha_1 - \frac{1}{2}\partial\psi_1\bar{\alpha}_2\alpha_1 - \frac{1}{4}\bar{\partial}\partial\phi\bar{\alpha}_2\alpha_2\alpha_1\right. \\
&\left.+\frac{1}{2}\bar{\partial}\phi\alpha_2 - \frac{1}{2}\alpha_2\bar{\alpha}_1\bar{\partial}\psi_1 + \frac{1}{2}\bar{\partial}\bar{\psi}_1\alpha_2\alpha_1 - \frac{1}{4}\bar{\partial}F\alpha_2\bar{\alpha}_1\alpha_1 - \frac{1}{4}\bar{\partial}\partial\phi\alpha_2\bar{\alpha}_2\alpha_1 + \frac{1}{4}\bar{\partial}\partial\psi_1\alpha_2\bar{\alpha}_1\bar{\alpha}_2\alpha_1\right) \\
&+\left(\bar{\psi}_1 + \frac{1}{2}F\bar{\alpha}_1 + \frac{1}{2}\partial\phi\bar{\alpha}_2 - \frac{1}{2}\bar{\partial}\bar{\psi}_1\alpha_2\bar{\alpha}_1 + \frac{1}{2}\partial\psi_1\bar{\alpha}_1\bar{\alpha}_2 - \frac{1}{4}\bar{\partial}\partial\phi\bar{\alpha}_2\alpha_2\bar{\alpha}_1\right. \\
&\left.-\frac{1}{2}\bar{\alpha}_2\partial\phi - \frac{1}{2}\partial\bar{\psi}_1\bar{\alpha}_2\alpha_1 + \frac{1}{2}\bar{\alpha}_2\bar{\alpha}_1\partial\psi_1 + \frac{1}{4}\partial F\bar{\alpha}_2\bar{\alpha}_1\alpha_1 - \frac{1}{4}\partial\bar{\partial}\phi\bar{\alpha}_2\bar{\alpha}_1\alpha_2 + \frac{1}{4}\partial\bar{\partial}\bar{\psi}_1\bar{\alpha}_2\alpha_2\bar{\alpha}_1\alpha_1\right)\varepsilon_1 \\
&+\bar{\varepsilon}_2\left(\frac{1}{2}\partial\phi\alpha_1 + \frac{1}{2}\partial\psi_1\bar{\alpha}_1\alpha_1 - \frac{1}{4}\bar{\partial}\partial\phi\alpha_2\bar{\alpha}_1\alpha_1 + \frac{1}{2}\alpha_1\partial\phi - \frac{1}{2}\alpha_1\bar{\alpha}_1\partial\psi_1 + \frac{1}{4}\partial\bar{\partial}\phi\alpha_1\bar{\alpha}_1\alpha_2\right) \\
&+\left(-\frac{1}{2}\bar{\partial}\phi\bar{\alpha}_1 - \frac{1}{2}\bar{\partial}\bar{\psi}_1\bar{\alpha}_1\alpha_1 - \frac{1}{4}\bar{\partial}\partial\phi\bar{\alpha}_2\bar{\alpha}_1\alpha_1 - \frac{1}{2}\bar{\alpha}_1\bar{\partial}\phi + \frac{1}{2}\bar{\alpha}_1\bar{\partial}\bar{\psi}_1\alpha_1 + \frac{1}{4}\bar{\alpha}_1\bar{\partial}\partial\phi\bar{\alpha}_2\alpha_1\right)\varepsilon_2
\end{aligned} \tag{E.6}$$

The covariant derivatives read

$$D_1 = -\frac{\partial}{\partial\bar{\alpha}_1} - \frac{1}{2}\alpha_2\bar{\partial}, \quad \bar{D}_1 = \frac{\partial}{\partial\alpha_1} + \frac{1}{2}\bar{\alpha}_2\partial, \quad D_2 = -\frac{\partial}{\partial\bar{\alpha}_2} - \frac{1}{2}\alpha_1\partial, \quad \bar{D}_2 = \frac{\partial}{\partial\alpha_2} + \frac{1}{2}\bar{\alpha}_1\bar{\partial}, \tag{E.7}$$

and the chiral superfield fulfils the constraint

$$D_2\Phi = \frac{1}{2}\partial\phi\alpha_1 + \frac{1}{2}\partial\psi_1\bar{\alpha}_1\alpha_1 - \frac{1}{4}\bar{\partial}\partial\phi\alpha_2\bar{\alpha}_1\alpha_1 - \frac{1}{2}\alpha_1\partial\phi + \frac{1}{2}\alpha_1\bar{\alpha}_1\partial\psi_1 - \frac{1}{4}\partial\bar{\partial}\phi\alpha_1\bar{\alpha}_1\alpha_2 = 0 \tag{E.8}$$

$$\bar{D}_2\Phi = -\frac{1}{2}\bar{\partial}\phi\bar{\alpha}_1 - \frac{1}{2}\bar{\partial}\bar{\psi}_1\bar{\alpha}_1\alpha_1 - \frac{1}{4}\bar{\partial}\partial\phi\bar{\alpha}_2\bar{\alpha}_1\alpha_1 + \frac{1}{2}\bar{\alpha}_1\bar{\partial}\phi - \frac{1}{2}\bar{\alpha}_1\bar{\partial}\bar{\psi}_1\alpha_1 - \frac{1}{4}\bar{\alpha}_1\bar{\partial}\partial\phi\bar{\alpha}_2\alpha_1 = 0 \tag{E.9}$$

The antichiral field can be constructed in complete analogy

$$\bar{\Phi} = \exp(-\delta_\varepsilon)\bar{\phi} = \sum_{n=0}^{4}\frac{1}{n!}(-\delta_\varepsilon)^n\bar{\phi} \tag{E.10}$$

which yields

$$\begin{aligned}
\delta_\varepsilon\bar{\phi} =&\bar{\psi}_2\varepsilon_2 + \bar{\varepsilon}_2\psi_2 \\
\delta_\varepsilon^2\phi =&\delta\bar{\psi}_2\varepsilon_2 + \bar{\varepsilon}_2\delta\psi_2 = -\bar{F}\bar{\varepsilon}_2\varepsilon_2 - \bar{\partial}\bar{\phi}\bar{\varepsilon}_1\varepsilon_2 + \partial\bar{\phi}\bar{\varepsilon}_2\varepsilon_1 \\
\delta_\varepsilon^3\phi =&-\delta\bar{F}\bar{\varepsilon}_2\varepsilon_2 - \bar{\partial}\delta\bar{\phi}\bar{\varepsilon}_1\varepsilon_2 + \partial\delta\bar{\phi}\bar{\varepsilon}_2\varepsilon_1 = -3\partial\bar{\psi}_2\varepsilon_1\bar{\varepsilon}_2\varepsilon_2 + 3\bar{\partial}\psi_2\bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_2 \\
\delta_\varepsilon^4\phi =&-3\partial\delta\bar{\psi}_2\varepsilon_1\bar{\varepsilon}_2\varepsilon_2 + 3\bar{\partial}\delta\psi_2\bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_2 = 6\bar{\partial}\partial\bar{\phi}\bar{\varepsilon}_1\varepsilon_1\bar{\varepsilon}_2\varepsilon_2.
\end{aligned} \tag{E.11}$$

The superfield reads

$$\begin{aligned}\bar{\Phi} &= \bar{\phi} - (\bar{\psi}_2\alpha_2 + \bar{\alpha}_2\psi_2) + \frac{1}{2}(-\bar{F}\bar{\alpha}_2\alpha_2 - \bar{\partial}\bar{\phi}\bar{\alpha}_1\alpha_2 + \partial\bar{\phi}\bar{\alpha}_2\alpha_1) - \frac{1}{3!}(-3\partial\bar{\psi}_2\alpha_1\bar{\alpha}_2\alpha_2 + 3\bar{\partial}\psi_2\bar{\alpha}_1\bar{\alpha}_2\alpha_2) \\ &\quad + \frac{1}{4!}(6\bar{\partial}\partial\bar{\phi}\bar{\alpha}_1\alpha_1\bar{\alpha}_2\alpha_2) \\ &= \bar{\phi} - \bar{\psi}_2\alpha_2 - \bar{\alpha}_2\psi_2 - \frac{1}{2}\bar{F}\bar{\alpha}_2\alpha_2 - \frac{1}{2}\bar{\partial}\bar{\phi}\bar{\alpha}_1\alpha_2 + \frac{1}{2}\partial\bar{\phi}\bar{\alpha}_2\alpha_1 + \frac{1}{2}\partial\bar{\psi}_2\alpha_1\bar{\alpha}_2\alpha_2 - \frac{1}{2}\bar{\partial}\psi_2\bar{\alpha}_1\bar{\alpha}_2\alpha_2 \\ &\quad + \frac{1}{4}\bar{\partial}\partial\bar{\phi}\bar{\alpha}_1\alpha_1\bar{\alpha}_2\alpha_2. \end{aligned} \tag{E.12}$$

The kinetic term is given by

$$-2\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\mathrm{d}^2\alpha\,\mathrm{d}^2\bar{\alpha}\,\bar{\Phi}\Phi, \quad \mathrm{d}^2\alpha = \alpha_1\alpha_2, \quad \mathrm{d}^2\bar{\alpha} = \bar{\alpha}_2\bar{\alpha}_1 \tag{E.13}$$

and the potential term by

$$-2\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\mathrm{d}\bar{\alpha}_1\mathrm{d}\alpha_1 W(\Phi) - 2\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\mathrm{d}\bar{\alpha}_2\mathrm{d}\alpha_2 \bar{W}(\bar{\Phi}). \tag{E.14}$$

The most general supersymmetric action for a number of fields $\Phi^i, \bar{\Phi}^i$ with $i = 1, \ldots, n$ is given by

$$-\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\mathrm{d}^2\alpha\,\mathrm{d}^2\bar{\alpha}\,K(\bar{\Phi}^i\Phi^i) - 2\left(\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\mathrm{d}\bar{\alpha}_1\mathrm{d}\alpha_1 W(\Phi^i) + \mathrm{h.c.}\right) \tag{E.15}$$

where $K\left(\bar{\Phi}^i, \Phi^i\right)$ is a Kähler potential. The $\mathcal{N} = (2,2)$ model is obtained for $i = 1$ where K, W and $\bar{W}$ are arbitrary functions of the field.

E.2 Flow equation for the momentum-dependent wave-function renormalization

To obtain the flow equations for the wave-function renormalisation the second derivative of the effective action is decomposed into a field-independent part $\Gamma_0^{(2)} + R_k$ and a field-dependent part $\Delta\Gamma_k^{(2)}$. In the following we drop the momentum dependence of the regulators for simplicity of notation:

$$(\Gamma_0^{(2)} + R_k)(q, q') + \Delta\Gamma_k(q, q') = \begin{pmatrix} A_0 & 0 \\ 0 & B_0 \end{pmatrix} \delta(q - q') + \begin{pmatrix} \Delta A & \Delta C \\ \Delta D & \Delta B \end{pmatrix} \tag{E.16}$$

Recall that $h = (1 + r_2)Z_k^2(q)$, $M = (r_1 Z_k^2(q) + m)$. With this, the blocks read

$$A_0 = \begin{pmatrix} q^2 h \cdot \mathbb{1} & M \cdot \sigma_3 \\ M \cdot \sigma_3 & -h \cdot \mathbb{1} \end{pmatrix}, \quad B_0 = i\not{q}h + M\mathbb{1} \tag{E.17}$$

and

$$\Delta A = 2g \begin{pmatrix} F_1 & -F_2 & \phi_1 & -\phi_2 \\ -F_2 & -F_1 & -\phi_2 & -\phi_1 \\ \phi_1 & -\phi_2 & 0 & 0 \\ -\phi_2 & -\phi_1 & 0 & 0 \end{pmatrix} (q+q')\,,\ \Delta C = 2g \begin{pmatrix} \bar\psi_1 & i\bar\psi_2 \\ \bar\psi_1 & -i\bar\psi_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} (q+q')$$
$$\Delta D = 2g \begin{pmatrix} \psi_1 & i\psi_1 & 0 & 0 \\ \psi_2 & -i\psi_2 & 0 & 0 \end{pmatrix} (q+q')\,,\ \Delta B = 2g \begin{pmatrix} \phi_1 + i\phi_2 & 0 \\ 0 & \phi_1 - i\phi_2 \end{pmatrix} (q+q')\,. \tag{E.18}$$

The flow equation can then be expanded [74] in

$$\partial_t \Gamma_k = \frac{1}{2}\tilde\partial_t \,\mathrm{STr}\left((\Gamma_0^{(2)} + R_k)^{-1}\Delta\Gamma\right) - \frac{1}{4}\tilde\partial_t \,\mathrm{STr}\left((\Gamma_0^{(2)} + R_k)^{-1}\Delta\Gamma\right)^2 + \dots \tag{E.19}$$

with $\tilde\partial_t$ acting only on the regulator. STr denotes a trace in field space as well as an integration in momentum space. The wave-function renormalisation is a term proportional to F_i^2 and can be obtained from the second term in this expansion. To calculate this we define the abbreviations

$$M(q,q') \equiv \int_{q''} (\Gamma_0^{(2)} + R_k)^{-1}(q)\delta(q+q'')\Delta\Gamma(q'',q') = (\Gamma_0^{(2)} + R_k)^{-1}(q)\Delta\Gamma(-q,q'). \tag{E.20}$$

Then the second term in the expansion reads

$$\begin{aligned} &\tilde\partial_t \,\mathrm{Str} \int_{q,q'} M(q,q')M(q',q) \\ &= \mathrm{Str}\int_{q,q'} (\Gamma_0^{(2)} + R_k)^{-1}(q)\partial_t R_k (\Gamma_0^{(2)} + R_k)^{-1}(q)\Delta\Gamma(-q,q')(\Gamma_0^{(2)} + R_k)^{-1}(q')\Delta\Gamma(-q',q) \\ &+ \mathrm{Str}\int_{q,q'} (\Gamma_0^{(2)} + R_k)^{-1}(q)\Delta\Gamma(-q,q')(\Gamma_0^{(2)} + R_k)^{-1}(q')\partial_t R_k(q')(\Gamma_0^{(2)} + R_k)^{-1}(q')\Delta\Gamma(-q',q) \end{aligned} \tag{E.21}$$

where Str denotes a trace in field space. The functional derivative is taken with respect to $F_i(p)$ and $F_i(-p)$ and all fields are set to zero in order to project on the wave-function renormalisation $Z_k(p^2)$. This yields

$$\begin{aligned} \partial_k Z_k^2(p) = &-8g^2 \int \frac{d^2q}{4\pi^2} \frac{h(p-q)h(q)}{v(q)^2 v(p-q)^2} \left[\partial_k R_1(q-p)M(p-q)v(q) + \partial_k R_1(q)M(q)v(p-q)\right] \\ &+ 4g^2 \int \frac{d^2q}{4\pi^2} \frac{h(p-q)\partial_k R_2(q)u(q)v(p-q)}{v(q)^2 v(p-q)^2} + 4g^2 \int \frac{d^2q}{4\pi^2} \frac{h(q)\partial_k R_2(q-p)v(q)u(p-q)}{v(q)^2 v(p-q)^2} \end{aligned} \tag{E.22}$$

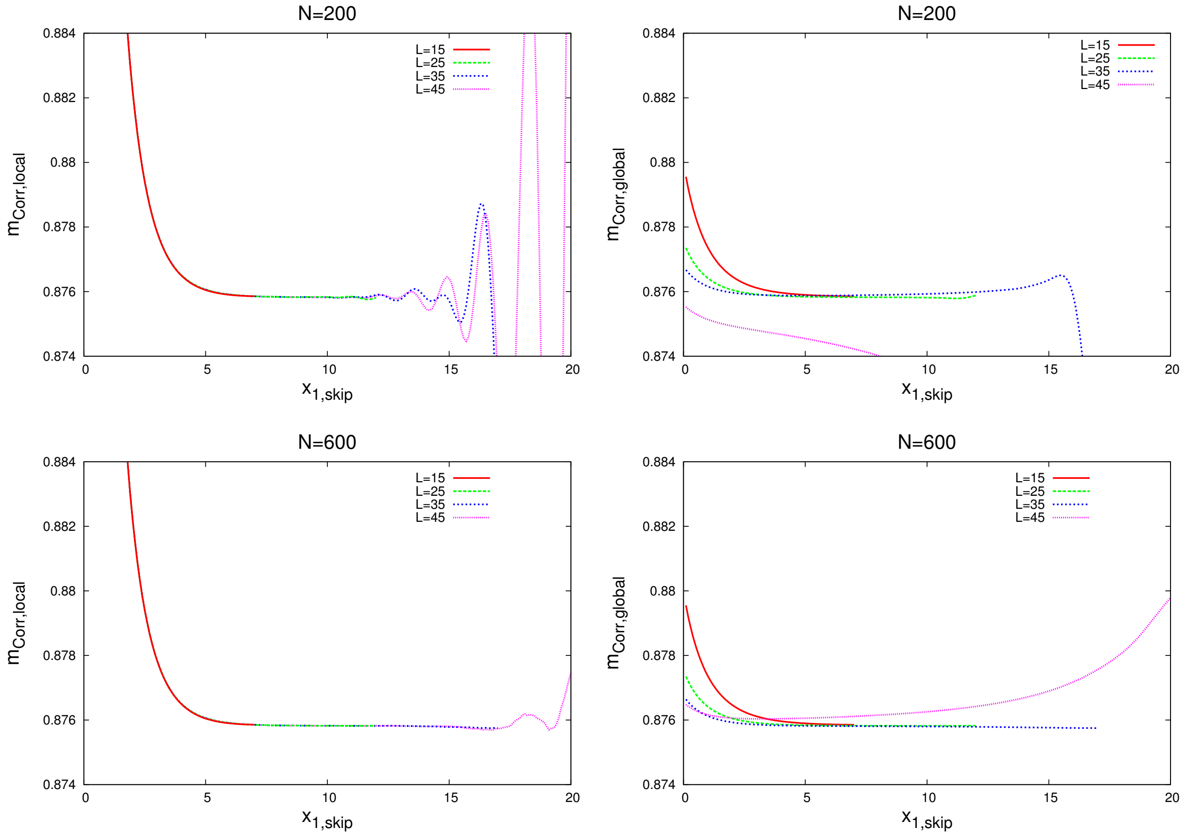


Figure E.1: *Left panel:* $m_{\text{corr}}^{\text{local}}$ with $\lambda = 0.6$ for discretizations $N = 200$ and $N = 600$ and different box sizes $L = 15, 25, 35$ and 45. *Right panel:* $m_{\text{corr}}^{\text{global}}$ with $\lambda = 0.6$ for discretizations $N = 200$ and $N = 600$ and different box sizes $L = 15, 25, 35$ and 45.

with the abbreviations

$$h(q) = \left(r_2\left(k,q\right)+1\right) Z_k^2\left(q\right),\; M(q) = m + r_1(k,q) Z_k^2\left(q\right),\; R_i(q) = r_i\left(k,q\right) Z_k^2\left(q\right), \qquad \text{(E.23)}$$
$$u(q) = M(q)^2 - q^2h^2(q),\; v(q) = M(q)^2 + q^2h^2(q)$$

E.3 Determination of the renormalized mass

The numerical calculations of Z_k^2 in the main text use a grid of $N = 60$ points in the direction of p^2, distributed equidistantly on a logarithmic scale. The result for $Z_{k\to 0}(p)$ is interpolated with splines to calculate the propagator $G_{\text{bos}}^{\text{NLO}}(p)$. A discrete Fourier transformation of $G_{\text{bos}}^{\text{NLO}}(p)$ yields the correlator $C(x_1)$ on the interval $x_1 \in [0, L]$ with $n = 10\,001$ intermediate points. In the main text we use $L = 15$. From its large distance behaviour

$$C_{a,m_{\text{cor}}}(x_1) \propto \cosh\left(m_{\text{corr}}\left(x_1 - \frac{L}{2}\right)\right) \qquad \text{(E.24)}$$

the correlator mass m_{corr} is determined by a least square fit. The fit range is constrained to the interval $[x_{1,skip}, \ldots, L - x_{1,skip}]$ where the contributions of excited states are negligible. The value of $x_{1,skip}$ is determined such that $m_{\text{corr}}(x_{1,skip})$ shows a plateau. Either the fit is made on the whole range $[x_{1,skip}, \ldots, L - x_{1,skip}]$ – this quantity is called $m_{\text{corr}}^{\text{global}}$ – or just inside a small interval of size 0.2 starting from $x_{1,\text{skip}}$ – this quantity is called $m_{\text{corr}}^{\text{local}}$.

In the left panel of figure E.1 $m_{\text{corr}}^{\text{local}}$ is shown for two different discretisations of $Z_k^2(p^2)$, $N = 200$ in the upper and $N = 600$ in the lower panel. In the right panel the same is shown for $m_{\text{corr}}^{\text{global}}$. From these plots we can read off that for $x_{1,skip}$ not too large there is a clear plateau which is stable if the box size is increased. But for very large $x_{1,skip}$ the local mass oscillates. As this oscillation is reduced when the discretisation is increased it is due to fluctuations in the spline interpolation of Z_k^2. At small values of the correlator the numerical errors are more important for the masses. As the fluctuations become visible for large box sizes, in these cases the global mass fit is of no use because it averages over the local mass and is strongly influenced by the oscillations. For this reason we take the plateau of $m_{\text{corr}}^{\text{local}}$ as the value of the renormalised mass.

F Diagrammatic description of the flow equation

In this section we introduce the diagrammatic representation of flow equations. The flow equation can be rewritten as

$$\partial_t \Gamma_k = \frac{1}{2} \,\mathrm{STr}\, \tilde{\partial}_t \ln \left(\Gamma_k^{(2)} + R_k \right)$$

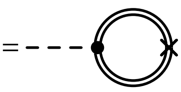
The superpotential is obtained by a projection on the terms linear in the auxiliary fields. The flow equation can be written as

$$\partial_t W_k'(\phi) = \frac{1}{2} \,\mathrm{STr} \left(\frac{1}{P}(\phi) \frac{\partial P(\phi)}{\partial F} \right) \Bigg|_{F=0,\bar{\psi}=\psi=0} .$$

For the diagrammatic notation we use the following symbols:

═══	$^1/_P$,
- - - -	auxiliary field line,
×	$^1/_2\, \tilde{\partial}_t$

The flow equation for the superpotential reads in diagrammatic notation:

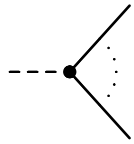
$\partial_t W_k' =$ 

Now it is straightforward to see why only terms that are at most proportional to F^3 can directly influence the flow of the superpotential: Terms proportional to F^n ($n \leq 3$) in the ansatz for the effective action correspond to vertices with n external auxiliary field lines. To contribute to the flow equation of the superpotential all but one auxiliary field lines have to be contracted. For $n > 3$ this is not possible due to the one-loop structure of the flow equation.

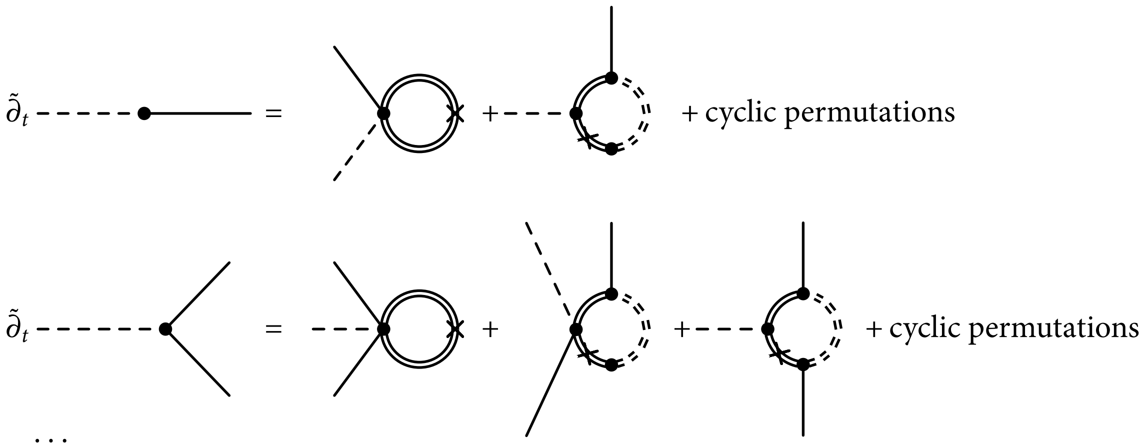
In a polynomial approximation of the superpotential,

$$W'(\phi) = \sum_{i=1}^{N} a_i \phi^i$$

the coupling a_i can be represented as

$j! \cdot a_j$: 

In the LPA the polynomially expanded flow equation evaluated at vanishing ϕ is represented as



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Zusammenfassung

Eine der wichtigsten Errungenschaften der theoretischen Physik des 20. Jahrhunderts ist das Standardmodell der Elementarteilchenphysik. Dieses Modell erlaubt, die Vielfalt der in Beschleunigerexperimenten beobachteten Elementarteilchen zu klassifizieren und ihre Eigenschaften und Wechselwirkungen zu beschreiben. Bis zu einer Energieskala von einigen 100 GeV sind die Vorhersagen des Standardmodells hervorragend bestätigt worden. Die Klassifikation der Elementarteilchen beruht auf der Ausnutzung von Symmetrien. Aus diesem Grund ist es naheliegend, sich für eine Erweiterung des Standardmodells, die für die Beschreibung der Physik bei größeren Energieskalen notwenig ist, ebenfalls auf (neue) Symmetrien zu stützen. Einen vielversprechenden Kandidaten bietet die Supersymmetrie, welche Bosonen und Fermionen, also Teilchen mit ganz- und halbzahligem Spin verknüpft.

Die vorliegende Arbeit beschäftigt sich mit der Anwendung der funktionalen Renormierungsgruppe (FRG) auf supersymmetrische Theorien. Dazu werden Skalarfeldtheorien in verschiedenen Dimensionen untersucht.

Um supersymmetrische Flussgleichungen zu erhalten, muss die Cutoffwirkung die Supersymmetrie respektieren. Dies wird erreicht, indem die Cutoffwirkung im Superraum formuliert wird. Um die Ein-Loop-Struktur zu erhalten, wird sie als Funktion quadratisch in den Superfeldern gewählt. Der Regulator ist eine Funktion der superkovarianten Ableitungen. Auch die Trunkierung muss so gewählt werden, dass sie die Supersymmetrie respektiert. Dies wird dadurch erreicht, dass die Wirkung ebenfalls im Superraum trunkiert wird. Für eine solche Trunkierung bietet sich eine Entwicklung in Superfeldern und superkovarianten Ableitungen an.

Auf Ebene der Komponenten entspricht eine solche Entwicklung gerade einer Entwicklung in Potenzen des Hilfsfeldes mit entsprechenden supersymmetrischen Partnertermen. Dies hat zur Folge, dass, im Vergleich zu einer reinen Ableitungsentwicklung, verschiedene Impulsordnungen gemischt werden und z. B. die Wellenfunktionsrenormierung einen viel größeren Einfluss hat als in einer nichtsupersymmetrischen Theorie. Bei größeren Kopplungen gewinnen auch die höheren Ordnungen des Hilfsfeldes einen immer größeren Einfluss auf die quantitativen Ergebnisse.

Supersymmetrie führt außerdem zu einer engen Verflechtung von bosonischen und fermionischen Regulatoren und erzwingt eine Regulatorstruktur, welche von der aus Theorien mit Yukawa-Wechselwirkungen ohne Supersymmetrie benutzten abweicht. Insbesondere erzwingt die Supersymmetrie eine Regularisierung des Hilfsfeldes.

Da in jeder Ordnung der Trunkierung höhere Potenzen des Hilfsfeldes eingeführt werden, kann der Fluss für die interessierenden Größen sehr einfach durch Projektion auf die entsprechenden Potenzen des Hilfsfeldes abgeleitet werden, da diese keine Impulspotenzen enthalten.

Den Anfang der Arbeit bildet die Untersuchung einer $0 + 1$ dimensionalen Feldtheorie, der supersymmetrischen Quantenmechanik, welche als eindimensionales Wess-Zumino Modell interpretiert werden kann. Es wird der Fall der ungebrochenen Supersymmetrie betrachtet. An diesem Modell wird die Konstruktion einer supersymmetrischen Cutoffwirkung demonstriert und die erforderliche Regulatorstruktur abgeleitet. Die erste angeregte Energie lässt sich durch Diagonalisierung der Hamiltonfunktion berechnen, was zuverlässige Vergleichswerte liefert. Ein Vergleich mit den Ergebnissen aus der FRG-Rechnung zeigt, dass schon für kleine Kopplungen die Wellenfunktionsrenormierung berücksichtigt werden muss, um eine quantitative Übereinstimmung mit den Ergebnissen aus der Diagonalisierung zu erreichen. Werden die Kopplungen so groß, dass das Superpotential an der Cutoffscala nicht mehr konvex ist, bricht die Näherung zusammen. In diesem Parameterbereich ist eine höhere Trunkierung notwendig, was die Hinzunahme von höheren Potenzen im Hilfsfeld und deren supersymmetrischen Partnertermen entspricht.

Die an diesem Modell gewonnenen Ergebnisse werden anschließend auf das $\mathcal{N} = 1$ Wess-Zumino Modell erweitert und angewendet. Interessant ist dieses Modell, weil es spontane Supersymmetriebrechung zeigt. Diese geht für die betrachteten Wess-Zumino Modelle einher mit einer Wiederherstellung der $\mathbb{Z}_2$-Symmetrie.

Die Supersymmetriebrechung kann im Kontext von kritischen Phänomenen verstanden werden, da die Phasengrenze zwischen supersymmetrisch gebrochener und ungebrochener Phase durch Feintuning der infrarotinstabilen Richtungen auf einen kritischen Punkt erreicht wird. Mit Hilfe der FRG wird die Fixpunktstruktur und die kritischen Exponenten des Modells untersucht. Das Modell hat unendliche viele Fixpunkte. Einer dieser Fixpunkte hat nur eine infrarotinstabile Richtung und ist ein Attraktor für alle Trajektorien der Flussgleichung. Der zu diesem Fixpunkt gehörige kritische Punkt bestimmt den Phasenübergang zwischen gebrochener und ungebrochener Supersymmetrie.

Für den kritischen Exponenten, der zu der einen, infrarotinstabilen Richtung gehört, ergibt sich ein direkter Zusammenhang zur anomalen Dimension, beide sind durch eine Skalenrelation verknüpft. Diese heißt *Superskalenrelation*, da sie nur in diesen supersymmetrischen Theorien und nicht in bosonischen Ising-Modellen auftritt. Diese Relation führt dazu, dass das Minimum des dimensionsbehafteten Potentials ausfriert, dass das Minimum also im Limes $k \to 0$ gegen einen konstanten Wert konvergiert. Außerdem bewirkt die Skalenrelation, dass die Masse des Skalarfeldes durch die RG-Skala bestimmt wird. Im Limes $k \to 0$ wird das Skalarfeld masselos.

Da das zweidimensionale Skalarfeld dimensionslos ist, stellt die die dimensionslose Flussgleichung einen Sonderfall dar. Insbesondere hat dies zur Folge, dass bei einer Fixpunktanalyse im Rahmen der niedrigsten Ordnung der superkovarianten Ableitungsentwicklung (LPA) nur ein Kontinuum von oszillierenden oder divergenten Lösungen zugänglich ist. Erst in der nächsten Ordnung (NLO), in der eine Wellenfunktionsrenormierung berücksichtigt wird, treten Fixpunktlösungen auf, die sich für große Felder polynomial im Außenbereich verhalten.

In den betrachteten Approximationen LPA und NLO ist das aus dem (konvexen) Superpotential nach Ausintegration des Hilfsfeldes berechnete Potential für das Skalarfeld in der Phase mit ungebrochener Supersymmetrie nicht konvex. Dies ist eine Konsequenz daraus, dass in einer supersymmetrischen Entwicklung verschiedene Impulspotenzen gemischt werden und deswegen in der Formulierung ohne Hilfsfelder nicht alle Beiträge mit verschwindendem Impuls zur Flussgleichung des Skalarfeldpotentials berücksichtigt werden. Um ein konvexes Potential für das Skalarfeld zu erhalten, muss ein Potential für die Hilfsfelder berücksichtigt werden. Dies gilt auch für die Modelle in höheren Dimensionen.

Formal sehr ähnlich zum zweidimensionalen Modell ist das $\mathcal{N} = 1$ Wess-Zumino Modell in drei Dimensionen. Aber ein wesentlicher Unterschied ist, dass das Skalarfeld in drei Dimensionen dimensionsbehaftet ist. Dies führt dazu, dass auch schon in der LPA Fixpunktlösungen gefunden werden, die sich für große Felder polynomial verhalten. Im dreidimensionalen Modell gibt es, neben dem trivialen Gaußschen Fixpunkt, nur einen weiteren nichttrivialen Fixpunkt. Dieser besitzt eine Richtung, die infrarotinstabil ist. Das Fixpunktpotential für das Skalarfeld, das sich nach Ausintegration des Hilfsfeldes ergibt, hat im Außenbereich ein Verhalten wie ϕ^6, es ist also gerechtfertigt, diesen Fixpunkt als das supersymmetrische Analogon des Wilson-Fischer Fixpunktes in dreidimensionalen, isingartigen Theorien zu betrachten.

Auch im dreidimensionalen Modell gibt es eine Skalenrelation zwischen dem kritischen Exponenten der instabilen Richtung und der anomalen Dimension, die der Relation in zwei Dimensionen formal ähnlich ist. Auch im dreidimensionalen Modell bewirkt sie ein Ausfrieren des dimensionsbehafteten Potentials. Sie hat ebenfalls zur Konsequenz, dass die Masse des Skalarfeldes auch in drei Dimensionen durch die RG-Skala bestimmt wird und das Skalarfeld im Infrarotlimes masselos wird. Das Modell wurde außerdem bei endlichen Temperaturen untersucht. Für ein Gas aus masselosen Skalarfeldern ist zu erwarten, dass es dem Stefan-Boltzmann Gesetz in $2 + 1$ Dimensionen genügt. Dies konnte für das gegebene Modell im wesentlichen bestätigt werden. Desweiteren konnte das Phasendiagramm, bezogen auf die Wiederherstellung der $\mathbb{Z}_2$-Symmetrie, für endliche Temperaturen berechnet werden. Für jeden Parameterwert, bei dem die $\mathbb{Z}_2$-Symmetrie gebrochen ist, gibt es eine kritische Temperatur, bei der die Symmetrie wieder hergestellt wird.

Den Abschluss der Arbeit bildet die Untersuchung des zweidimensionalen $\mathcal{N} = (2, 2)$ Wess-Zumino Modells. Dieses Modell wird aus der Dimensionsreduktion des vierdimensionalen $\mathcal{N} = 1$ Modells gewonnen. Es hat viele Eigenschaften des vierdimensionalen Modells, so kann z. B. keine Supersymmetriebrechung auftreten und das holomorphe Superpotential unterliegt einem Nichtrenormierungstheorem, d. h. die nackten Größen im Superpotential werden nicht renormiert.

Die physikalische Masse wird renormiert, bedingt unter anderem durch die Wellenfunktionsrenormierung. Die renormierte Masse kann über die Berechnung der Wellenfunktionsrenormierung mit Hilfe der FRG bestimmt werden. Diese Werte lassen sich mit Resultaten aus Monte-Carlo Simulationen vergleichen. Dieser Vergleich ist direkt möglich, da das Modell in zwei Dimensionen dank des Nichtrenormierungstheorems endlich ist.

Für kleine Kopplungen im Bereich, in dem auch Störungstheorie gültig ist, ist die Übereinstimmung zwischen FRG und Monte-Carlo Rechnung sehr gut, wenn in der FRG-Rechnung eine Impulsabhängigkeit der Wellenfunktionsrenormierung berücksichtigt wird. Ohne Impulsabhängigkeit gibt es deutliche Abweichungen zu den Gitterresultaten. Im Bereich mittlerer Kopplungsstärken wird die Übereinstimmung deutlich schlechter, in diesem Bereich gewinnen Operatoren höherer Ordnung in der superkovarianten Entwicklung an Bedeutung. Um in diesem Bereich eine Übereinstimmung mit den Gitterresultaten zu erzielen, muss die Trunkierung des Ansatzes für die Wirkung erweitert werden.

Diese Ergebnisse zeigen, dass die Wellenfunktionsrenormierung alleine – auch mit Impulsabhängigkeit – außerhalb des Bereiches, in dem Störungstheorie gültig ist, nicht ausreicht, um die vollen Quanteneffekte, die zur Renormierung der physikalischen Masse führen, zu berücksichtigen.

Abschließend lässt sich sagen, dass die funktionale Renormierungsgruppe so erweitert werden kann, dass sie auf supersymmetrische Theorien anwendbar ist und für diese Theorien quantitative Aussagen liefert. Allerdings erzwingt die Erhaltung der Supersymmetrie eine Mischung verschiedener Impulspotenzen in der trunkierten Wirkung. Dies führt dazu, dass dass insbesondere im Bereich größerer Kopplungsstärken höhere Ordnungen in der Trunkierung benötigt werden. Es müssen also höhere Potenzen des Hilfsfeldes mit ihren supersymmetrischen Partnern berücksichtigt werden.

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Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet. Ergebnisse, die in Zusammenarbeit mit den Mitgliedern des Lehrstuhles für Quantenfeldtheorie in Jena und anderen Kooperationen entstanden sind, sind in der Arbeit entsprechend benannt.

Weitere Personen waren an der inhaltlich-materiellen Erstellung der vorliegenden Arbeit nicht beteiligt. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs- bzw. Beratungsdiensten (Promotionsberater und andere Personen) in Anspruch genommen. Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen.

Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die geltende Promotionsordnung der Physikalisch-Astronomischen Fakultät ist mir bekannt.

Ich versichere ehrenwörtlich, dass ich nach bestem Wissen die reine Wahrheit gesagt und nichts verschwiegen habe.

Jena, January 16, 2011

Franziska Synatschke-Czerwonka

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