Domain Walls and Special-Holonomy Manifolds in String and M Theory

Dissertation

zur Erlangung des akademischen Grades

doctor rerum naturalium (Dr. rer. nat.)

vorgelegt dem Rat der Physikalisch-Astromomischen Fakultät
der Friedrich-Schiller-Universität Jena

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geboren am 29. Juli 1975 in Schwäbisch Gmünd

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Tag der letzten Rigorosumsprüfung:	24. September 2004
Tag der öffentlichen Verteidigung:	9. November 2004

Contents

I Introduction

A String Theory

String theory [1, 2] tries to unify all known interactions. Instead of many different point particles which sweep out world-lines, one has a *single* fundamental string which sweeps out a world-sheet. An important ingredient is supersymmetry, since it guarantees stability, *e.g.* absence of tachyons. There are five consistent string theories in ten dimensions. The low-energy approximation of these string theories is captured by five supergravity theories.

All these theories are related to one another by dualities, and there is evidence that they are really different expressions of a *single* theory, called M theory [3]. It should be said that up to now no microscopic description of M theory is known, in contrast to the world-sheet description of string theory. However, it has been argued that the unique supergravity theory in eleven dimensions is the low-energy approximation of M theory. Eleven-dimensional supergravity is related to one of the ten-dimensional supergravity theories (type-IIA) by a Kaluza-Klein reduction on a circle. Conversely, the strong-coupling limit of type-IIA string theory is eleven-dimensional supergravity.

String and M theory exist in ten and eleven dimensions, but our world has four space-time dimensions. Therefore, one has to compactify string theory from ten to four dimensions on suitable compact internal manifolds in order to obtain 'realistic' models. The four-dimensional fields arise from fluctuations around a given background. By this dimensional reduction, however, the predictive power of string and M theory is diminished, since there are many choices for the internal manifolds. This problem has two aspects. First, up to now it is not known how to decide which are the 'correct' internal manifolds on which to perform the compactification. Second, the four-dimensional couplings and masses are in general not fixed by the dimensional reduction. At the level of four-dimensional supergravity actions this means that there are flat directions in the potential for the scalar fields (moduli) which determine the parameters of the theory. This is the 'moduli degeneration problem'.

Starting with the early work of Strominger [4], in the last years new backgrounds of string theory have attracted quite some attention [5, 6, 7, 8, 9, 10, 11], because these types of backgrounds can improve on the 'moduli degeneration problem' of string theory. In these new backgrounds also p-form gauge fields are switched on. This leads to a back-reaction which deforms the geometry, *e.g.* from Ricci-flat $SU(3)$-holonomy manifolds to $SU(3)$-structure manifolds with torsion. The associated four-dimensional effective actions are *gauged* supergravity theories, which means that some scalar fields are charged under the vector fields of the theory. Moreover, there is a potential for the scalar fields, and as a consequence some or even all flat directions are lifted.

Another fascinating aspect of string theory is that there are mechanisms which avoid singularities. Of course, it is hoped that ultimately string and M theory are able to resolve all singularities, in particular the cosmological and black-hole singularities. While these goals have not been achieved in general, a variety of insights into the problem of singularities has already been found. One new ingredient is the existence of a length scale, $\sqrt{\alpha'}$, at which internal string states can be excited. This leads to a very soft UV behaviour of scattering amplitudes, which reflects itself in an infinite series of higher derivative terms in the low-energy effective action and in particular in higher curvature terms.

Besides higher derivative terms, there is another generic mechanism for avoiding singularities in string and M theory, which one might call "the intervention of additional states." One example of this mechanism are twisted states in toroidal orbifold compactifications [12], which prevent the conic singularities of these spaces to cause singularities of observable quantities. More elaborate versions of the two basic mechanisms take care of the geometrical singularities occurring at special points in the moduli spaces of Calabi-Yau compactifications. Such special points are related to flop transitions [13], conifold singularities [14], conifold transitions [15] and more general extremal transitions [16, 17]. A further example of a mechanism which avoids singularities in string theory is the so-called enhançon mechanism [18]. Here, one considers certain space-time geometries which have a naked curvature singularity in the supergravity approximation. However, by considering the full string theory one realises that before the singularity can be reached, particular modes of branes wrapped on internal cycles become light, and therefore must be taken into account. The resulting space-time geometry is then free of naked singularities. This mechanism, which has been first observed in a specific compactification with $\mathcal{N} = 4$ supersymmetry[1], seems to be quite generic.

This thesis is organised as follows. In this chapter we discuss the relevant properties of supergravity backgrounds. In particular, we describe $SU(3)$ structures on six-dimensional manifolds and G_2 structures on seven-dimensional manifolds.

Chapter II deals with properties of a class of five-dimensional supergravity backgrounds. We consider domain-wall solutions of gauged five-dimensional supergravity. While these solutions have been known for some time [19, 20, 21], here we analyse the appearance of curvature singularities. We prove that curvature singularities do not occur in precisely those models which have an interpretation as M theory flux-compactifications on non-singular Calabi-Yau manifolds. For this class of theories, we are able to establish the "enhançon-like" mechanism, which was found in particular examples in Ref. [22, 23], in a model-independent way.

In Chapter III, we construct two classes of backgrounds of four-dimensional gauged supergravity. Both classes of backgrounds are of domain-wall type. We generalise the domain walls of Ref. [24], which are the four-dimensional

[1] We count supersymmetry in multiples of the smallest spinor representation. In four dimensions $\mathcal{N} = 4$ corresponds to 16 real supercharges.

cousins of the domain walls we study in Chapter II, to domain walls with an arbitrary number of spectator hypermultiplets.[2] Furthermore, we construct a new class of domain walls, which are related to the compactification of type-IIA string theory on manifolds with a particular $SU(3)$ structure [25]. These so-called half-flat manifolds can be thought of as 'deformed' Calabi-Yau manifolds, and compared to Calabi-Yau manifolds these new backgrounds of string theory are much less understood. In particular their deformation theory is still largely unexplored. But new backgrounds of this type improve on the moduli-degeneration problem of string theory, since the four-dimensional supergravity theories are gauged. Using the four-dimensional domain-wall solutions, we are able to perform a nontrivial check of the proposal [25] of how to deform Calabi-Yau manifolds to half-flat manifolds. Moreover, we show explicitly that these four-dimensional domain walls can be lifted to ten-dimensional domain walls, and thus establish a consistent picture of this class of domain-wall solutions.

Chapter IV, finally, contains our conclusions and an outlook.

B Supersymmetric Backgrounds

Let Φ denote the fields of a given supergravity theory, in particular the metric, p-form gauge fields, and their fermionic superpartners such as the gravitino

[2] Note that in contrast to Ref. [24] we always consider *flat* domain walls.

and other spinors:

$$\Phi = \left\{ g_{MN},\ A_{M_1,\ldots,M_p},\ \psi_M^\alpha,\ \ldots \right\} . \tag{I.1}$$

Here, $M, N, \ldots$ denote space-time indices, while α is a spinor index. The condition on a supersymmetric background Φ_0 is that there exist supersymmetry parameters $\eta(x)$ in the given spinor representation, for which the supersymmetry variations vanish:

$$\delta_{\eta(x)} \Phi \Big|_{\Phi=\Phi_0} = 0 . \tag{I.2}$$

The number of spinors which solve (I.2) determine the degree to which Φ_0 is supersymmetric. We have, for instance, full supersymmetry if η is a spinor, which is not subject to additional algebraic conditions. If Φ_0 is fully supersymmetric, then the vanishing of the supersymmetry variations (I.2) imply that also the equations of motion are solved. This need not be the case if Φ_0 is only partially supersymmetric, where one has to demand that the equations of motion are fulfilled in addition to the supersymmetry conditions. For instance, the five and four-dimensional half-supersymmetric domain-wall solutions we consider in this thesis are $1/2$-BPS , which means that they are invariant under half of the supercharges only. In Chapter III, we consider a background of ten-dimensional type-IIA string theory which is a direct product of four-dimensional Minkowski space and a six-dimensional half-flat manifold. Although this is a supersymmetric background, the ten-dimensional equations of motion are not fulfilled since half-flat manifolds are not Ricci flat. Correspondingly, the equations of motion of the four-dimensional effective actions are not solved by Minkowski space. However, we are able to show that the gauged four-dimensional supergravity has domain-wall solutions which lift to ten-dimensional domain-wall solutions. The ten-dimensional background is then a direct product of three-dimensional Minkowski space and of a G_2-holonomy manifold with boundaries, which is Ricci flat.

Let us now expand on the construction of supersymmetric solutions of supergravity theories, which arise as low-energy approximations of string and M theory. In this thesis we consider bosonic backgrounds,

$$\Phi_0 = \left\{ \Phi_0^{\text{bos}} ,\ \Phi_0^{\text{fermion}} = 0 \right\} . \tag{I.3}$$

Since the bosons transform under a supersymmetry transformation into fermions, the variation of the bosonic fields vanish identically in a bosonic background. Thus, besides the equations of motion, the only nontrivial equations are

$$\delta_{\eta(x)} \Phi^{\text{fermion}} \big|_{\Phi=\Phi_0} = 0 . \tag{I.4}$$

Let us now consider the supersymmetry variation of the gravitino. In supergravity theories it is given by the super-covariant derivative of the supersymmetry parameter,

$$\delta_\eta \psi_M = \hat{D}_M^{s.c.} \big|_{\Phi_0^{bos}} \eta = 0 . \tag{I.5}$$

The supercovariant derivative, evaluated on a bosonic background, can be written schematically as

$$\hat{D}_M^{s.c.} \big|_{\Phi_0^{bos}} \eta = \Big(\nabla_M{}^{L.C.} + \text{``}\Big\{ \sum_p F_{M M_1 \ldots M_p} \Gamma^{M_1 \ldots M_p} \Big\}\text{''} \Big) \eta = 0 . \tag{I.6}$$

Hence, $\eta(x)$ has to be covariantly constant with respect to a 'generalised connection' [26]. Supersymmetric solutions can then be classified by the 'generalised holonomy' $Hol\big(\hat{D}_M\big) \subset GL\big(V_S\big)$ where $\eta(x) \in V_S$. In eleven-dimensional supergravity this group is contained in $GL(32, \mathbb{R})$ since η is a

Majorana spinor. Clearly, this group is bigger than the Riemannian holonomy group $SO(1,10)$. Another way to see that the holonomy of the generalised connection can be bigger than the Riemannian holonomy is that the part containing the p-form field strengths in (I.6) need not be of the form of a torsion contribution to the Levi-Civita (spin-)connection $\nabla_M^{L.C.}$. In this thesis we consider only backgrounds Φ_0 where the extra terms in the gravitino variation due to background gauge fields can be written as a torsion contribution to the Levi-Civita connection:

$$\hat{D}_M^{s.c.}\big|_{\Phi_0^{bos}}\eta = \Big(\nabla^{L.C.} + \nabla^T\Big)\eta = 0 \qquad \text{(I.7)}$$

Thus, we are in the realm of special holonomy manifolds [27].

By equation (I.7) the supersymmetry parameter η has to be covariantly constant with respect to the connection $\nabla^{L.C.} + \nabla^T$. Therefore we can normalize η point-wise,

$$\bar{\eta}(x)\eta(x) = 1\ , \quad \forall x \in M\ . \qquad \text{(I.8)}$$

In particular supersymmetry implies the existence of nowhere vanishing spinors.

If the background allows for k nowhere vanishing spinors, η^Λ, $\Lambda = 1\dots k$, we can build differential forms from spinor bilinears

$$\Omega^{\Lambda\Sigma|(p)} = \frac{1}{p!}\left(\bar{\eta}^\Lambda\,\gamma_{a_1a_2\dots a_p}\,\eta^\Sigma\right)\,e^{a_1}\wedge e^{a_2}\wedge\dots\wedge e^{a_p}\ . \qquad \text{(I.9)}$$

Here, e^a is a local basis of sections of the cotangent bundle T^*M, γ_a are gamma matrices, $\gamma_a\gamma_b + \gamma_b\gamma_a = 2\,\eta_{ab}$, and the spinors η^Λ are in the given spinor representation on M.

Some of these differential forms will vanish identically by symmetry properties of spinor bilinears. The others provide a set of nowhere vanishing differential forms on M. The existence of such differential forms, or, equivalently, of nowhere vanishing spinors, on a manifold M leads to a reduction of the structure group of TM, as we explain in the next section.

C From nowhere vanishing spinors to G-structures

In the mathematical literature on G-structures both the tangent bundle TM, and its associated frame bundle F^{TM} are used. This has the advantage that one can make use of vector bundle and of principal bundle technology. Let us introduce the notion of a frame bundle F^E associated to a vector bundle E.

There is a canonical way to associate a principal bundle to a vector bundle. Let $E \to M$ be a vector bundle of rank k over the manifold M. This means that the fibers of E are vector spaces $\mathbb{R}^k$ for which we choose a basis $\{e^a\}$, $a = 1 \ldots k$.

The associated principal bundle is called frame bundle F^E. As a manifold it has coordinates $(x, e^1, \ldots, e^k)$ with $x \in M$. The projection π: $F^E \to M$ is defined as

$$(x, e^1, \ldots, e^k) \mapsto x \ , \qquad \text{(I.10)}$$

and there is a $GL(k,\mathbb{R})$ action

$$g \cdot (x, e^1, \dots, e^k) = (x, e'^1, \dots, e'^k) \,, \quad e'^i = A^i{}_j e^j \tag{I.11}$$

with $g \in GL(k,\mathbb{R})$ and $A^i{}_j$ the natural representation of $GL(k,\mathbb{R})$ on the fiber $\mathbb{R}^k$ of E.

This construction associates to every vector bundle $E \to M$ a principal $GL(k,\mathbb{R})$-bundle $F^E \to M$. There is also an inverse construction which gives a 1-to-1 correspondence between vector bundles over M with fibre $\mathbb{R}^k$ and principal $GL(k,\mathbb{R})$ bundles. Note that principal bundles are more general than vector bundles, since their fibres can be also other groups than $GL(k,\mathbb{R})$.

In the following, we will consider the frame bundle associated with the tangent bundle of M: $F^{TM} =: F$. The structure group of this principal bundle is $GL(n,\mathbb{R})$, $n = \dim_{\mathbb{R}} M$.

A G-structure on M is a principal sub-bundle P of F with fibre $G \subset GL(n,\mathbb{R})$. By the correspondence between F^{TM} and TM this implies that the structure group of TM is reduced to G, which means that all transition functions lie in G. The existence of non-vanishing invariant tensor (or spinor) fields on M leads to a reduction of the structure group of TM to the subgroup of $GL(k,\mathbb{R})$ which leaves these tensor fields invariant. Let us mention the following examples.

If M is a Riemannian manifold, then there is a non-degenerate metric g

on M, which, at every point $x \in M$, can be brought to the form

$$g\big|_x = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \cdots + \mathrm{d}x^n \otimes \mathrm{d}x^n \ . \qquad \text{(I.12)}$$

This bilinear form is invariant under $O(n)$ rotations. Hence, the structure group of F can be at most $O(n) \subset GL(n, \mathbb{R})$. Clearly, in the context of supergravity solutions we are interested in Riemannian manifolds, because the metric is part of the background Φ_0.

Let now M be a Riemannian manifold of even dimension $n = 2m$. At each point we introduce complex coordinates by $\mathrm{d}z^i = \mathrm{d}x^i + i\mathrm{d}y^i$, $i = 1, \ldots, m$, in which the metric takes the form

$$g\big|_x = \mathrm{d}z^1 \otimes \mathrm{d}\bar{z}^1 + \mathrm{d}z^2 \otimes \mathrm{d}\bar{z}^2 + \cdots + \mathrm{d}z^m \otimes \mathrm{d}\bar{z}^m \ . \qquad \text{(I.13)}$$

If, furthermore, there is the following nowhere vanishing real two-form

$$J\big|_x = \frac{i}{2}\left(\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 + \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2 + \cdots + \mathrm{d}z^m \wedge \mathrm{d}\bar{z}^m\right) , \qquad \text{(I.14)}$$

we can build an almost complex structure $I_a^b := J_{ab}g^{cb}$ with the property $I^2 = -id$, which is compatible with the metric. The tangent bundle of (M, g, J) has an $U(m)$ structure, since the transformations which leave g and J invariant are at most $U(m) \subset O(2m)$ transformations. Such manifolds are called almost Hermitian. Here, it is crucial that the two-form J is nowhere vanishing, but precisely such differential forms are provided by the construction in the last section.

If, in addition to g and to J, there exists a nowhere vanishing m-form

$$\Omega\big|_x = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^m \ , \qquad \text{(I.15)}$$

the structure group of F is further reduced to $SU(m)$, since Ω transforms under an $U(m)$ transformation by multiplication with the determinant of the transformation. This form is of type $(m, 0)$ with respect to I. The forms J and Ω fulfill the equations

$$\begin{aligned} \underbrace{J \wedge J \wedge \cdots \wedge J}_{m \text{ times}} &= \frac{i^{(m+2)} m!}{2^m} \, \Omega \wedge \bar{\Omega} \, , \\ J \wedge \Omega &= 0 \, , \end{aligned} \tag{I.16}$$

which can be verified by using the explicit expressions given in Eqs. (I.14) and (I.15).

Let M_7 be a Riemannian manifold of (real) dimension 7. The subgroup of $O(7)$ which leaves the three-form

$$\varphi = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356} \tag{I.17}$$

invariant is the exceptional Lie group G_2. Here, $\mathrm{d}x^{a_1 \ldots a_p} := \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_p}$. The group G_2 also fixes the four-form

$$\star\varphi = \mathrm{d}x^{4567} + \mathrm{d}x^{2367} + \mathrm{d}x^{2345} + \mathrm{d}x^{1357} - \mathrm{d}x^{1346} - \mathrm{d}x^{1256} - \mathrm{d}x^{1247} \, , \tag{I.18}$$

the metric g, and the orientation on M_7 [27]. The three-form φ fulfills the following algebraic equations [28, 29]:

$$\varphi_{abc} \varphi^{cde} = 2\delta_{ab}^{[de]} + (\star\varphi)_{ab}{}^{de} \, . \tag{I.19}$$

Three-forms which are related to φ as defined in Eq. (I.17) by diffeomorphisms are called positive [27]. Hence, Eq. (I.19) is a more intrinsic way of saying that φ is positive. Every generic, positive three-form defines a G_2-structure on a seven-manifold [27].

D Holonomy

The parallel transport of a vector $V \in T_x M$ from a point $x \in M$ to an infinitesimally near point $x + \delta x$ is given by the vector $V^M(x) + \delta V^M(x)$ in $T_{x+\delta x} M$ defined as

$$\delta V^M := \Gamma^M_{NP} V^N \delta x^P \ . \tag{I.20}$$

The coefficients Γ^M_{NP} determine a connection ∇ in TM. Since TM is related to F^{TM}, there is a 1-to-1 correspondence between connections ∇ on TM and connections D in F. A connection D in a principal bundle P is a horizontal sub-bundle of TP. At any point $p \in P$ the connection D induces the split $TP_p = C_p \oplus D_p$ into a horizontal part D_p and into its complement C_p. A connection ∇ on TM is called compatible with a G-structure P on F, if the corresponding connection on F reduces to a connection on P. This means that at each point $p \in P$ $D_p \subset TF_p$ lies in TP_p.

Let γ be a path in M from the point x to the point y,

$$\gamma : \quad [0,1] \longrightarrow M \ , \quad \gamma(0) = x \ , \quad \gamma(1) = y \ . \tag{I.21}$$

Since parallel transport is governed by a first order differential equation, which is solved by (I.20), the parallel transport of a given a vector $V(0) \in T_x M$ along γ determines a unique vector $V(1) \in T_y M$. This map is linear,

$$V(1) = A_\gamma \cdot V(0) \ . \tag{I.22}$$

Since composition of (composable) paths translates into composition of maps, $A_{\gamma_1\gamma_2} = A_{\gamma_1} A_{\gamma_2}$, and since to every element there exists an inverse

$A_\gamma^{-1} = A_{\gamma^{-1}}$, the parallel transport map A_γ is an element of $GL(T_xM \to T_yM) \simeq GL(n, \mathbb{R})$. The parallel transport maps form the fundamental groupoid of M.

The Holonomy group of ∇ at a point $x \in M$ is defined as

$$Hol_x(\nabla) = \left\{A_\gamma \,\middle|\, \gamma(0) = \gamma(1) = x\right\} \subset GL(n, \mathbb{R}) \ . \tag{I.23}$$

This group is independent of the base point x in the sense that the groups $Hol_x(\nabla)$ and $Hol_y(\nabla)$ are conjugated with respect to the parallel transport map from x to y.

The following two statements are important later-on: [27]

- Given a connection ∇ on TM there is a compatible G-structure P on F if and only if $Hol(\nabla) \subset G$.
- Given a G-structure P on M and a connection D on F, there is always a connection on TM with $Hol(\nabla) \subset G$.

The next section deals with the question when there exist torsion-free connections ∇ on TM with $Hol(\nabla) \subset G$.

E Intrinsic torsion

The torsion tensor of a connection ∇ in TM is defined as

$$T(\nabla)^P_{MN} := \Gamma^P_{MN} - \Gamma^P_{NM} \ . \tag{I.24}$$

The intrinsic torsion $T^0(\nabla)$ is the obstruction to finding a torsion-free connection on a G-structure manifold M. It is called intrinsic, since it does not depend on a given connection, but only on the choice of G-structure on M. For a discussion of T^0 for general G-structures we refer to [27].

Let us first present an example. If M is an even-dimensional manifold, $\dim M = 2m$ with an almost complex structure I, then it can be shown that this almost complex structure is equivalent to a $GL(m, \mathbb{C})$-structure, and that I is a complex structure if the $GL(m, \mathbb{C})$-structure is torsion-free [27]. The intrinsic torsion is given by the Nienhius tensor.

Now we turn to G-structures on Riemannian manifolds, since we consider backgrounds of super*gravity* theories. As we have shown above the existence of a non-degenerate metric on a manifold reduces the G-structure to a subgroup of $O(n)$. In addition we fix an orientation on the manifold, since we consider only orientable manifolds in this thesis. This further reduces the structure group to $G \subset SO(n)$. The metric

$$g_{MN} = e^a_M e^b_N \, \delta_{ab} \tag{I.25}$$

can be expressed in terms of vielbeins

$$e^a := e^a_M \mathrm{d}x^M \ , \tag{I.26}$$

where $a, b, \ldots$ are flat indices and $M, N, \ldots$ are curved indices. The metric is compatible with a given connection ∇, if

$$0 = \nabla_N e^a_M = \partial_N e^a_M - \Gamma^P_{MN} e^a_P \ . \tag{I.27}$$

We obtain

$$\Gamma^M_{NP} = e^M_a \partial_P e^a_N \tag{I.28}$$

using that the vielbeins are invertible. Hence, on Riemannian manifolds a connection ∇ can be specified by providing a set of parallel frames e^a.

On a manifold M with an $SO(n)$-structure there is a unique torsion-free and metric compatible connection, the Levi-Civita connection $\nabla^{L.C.}$. In other words, given an arbitrary (metric compatible) connection ∇ on a Riemannian manifold, which generically will have torsion, one can always deform it to a torsion-free connection by local $SO(n)$ frame rotations.

$$e^a \to \Lambda^a{}_b(x) e^b \ , \quad \Lambda^a{}_b(x) \in SO(n) \ . \tag{I.29}$$

However, this procedure can fail if the structure group of TM is a proper subgroup $G \subset SO(n)$, since then only local G rotations of frames are allowed. Hence, $\nabla = \nabla^{L.C.} + \nabla^T$ and

$$T(\nabla)\Big|_x \ \in \ \mathfrak{so}(n) \otimes T^*_x M \tag{I.30}$$

with $\mathfrak{so}(n)$ the Lie algebra of the group $SO(n)$. Suppose now that $Hol(\nabla) \subset G \subset SO(n)$. Accordingly $\mathfrak{so}(n)$ splits into the Lie algebra $\mathfrak{g}$ of G and its complement,

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp \ . \tag{I.31}$$

The intrinsic torsion T^0 is defined as the projection of $T(\nabla)_x$ onto $\mathfrak{g}^\perp \otimes T^*_x M$,

$$T^0(\nabla)\Big|_x \in \mathfrak{g}^\perp \otimes T^*_x M \ . \tag{I.32}$$

It measures the failure of the Levi-Civita connection to have G-holonomy.

The intrinsic torsion of a given connection ∇ can be calculated by taking the covariant derivative of an invariant spinor (or invariant tensor fields):

$$\nabla \eta = \left(\nabla^{L.C.} + \nabla^{\mathfrak{g}^\perp} + \nabla^{\mathfrak{g}}\right)\eta = \left(\nabla^{L.C.} + \nabla^{\mathfrak{g}^\perp}\right)\eta \ . \tag{I.33}$$

The piece proportional to $\mathfrak{g}$ drops out, because the spinor is G-invariant, and what remains is the sum of the Levi-Civita connection and the intrinsic torsion.

Now, we introduce the intrinsic torsion of $SU(3)$ and of G_2 structures in preparation for Chapter III.

E.1 $SU(3)$ structure

An $SU(3)$ structure on a real 6-dimensional manifold is determined by specifying a nowhere vanishing two-form J and a three-form Ω satisfying (I.16)

$$\begin{aligned} J \wedge J \wedge J &= \tfrac{3i}{4}\, \Omega \wedge \bar{\Omega}\ , \\ J \wedge \Omega &= 0\ . \end{aligned} \tag{I.34}$$

The forms J and Ω also determine a metric g on M and the orientation [27, 30]. Note that if J and Ω are integrable, this statement is closely related to the Calabi-Yau theorem. The intrinsic torsion T^0 is an element of

$$T^0\big|_x \ \in\ \mathfrak{su}(3)^\perp \otimes T^*_x M \tag{I.35}$$

with the decomposition $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathfrak{su}(3)^\perp$. The intrinsic torsion can be computed by adapting Eq. (I.33) to the invariant forms J and Ω, that is by taking covariant exterior derivative, $\mathrm{d}_\nabla := alt(\nabla)$, of J and of Ω

$$\begin{aligned} \mathrm{d}_\nabla J = 0 \qquad &\Leftrightarrow \qquad \mathrm{d}J_{mnp} = 6\, T^0_{[mn}{}^r J_{r|p]}\ , \\ \mathrm{d}_\nabla \Omega = 0 \qquad &\Leftrightarrow \qquad \mathrm{d}\Omega_{mnpq} = 12\, T^0_{[mn}{}^r \Omega_{r|pq]}\ . \end{aligned} \tag{I.36}$$

By decomposing T^0 into irreducible representations of $\mathfrak{su}(3)$ one obtains five modules

$$T^0 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \tag{I.37}$$

of (real) dimension 2, 16, 12, 6, and 6 [31]. Let us now denote by W_i the components of T^0 in $\mathcal{W}_i$. Given J and Ω, the components $W_i \in \mathcal{W}_i$ can be obtained from Eq. (I.36):

$$\begin{aligned} \mathrm{d}J &= \tfrac{3i}{4}\big(W_1\bar{\Omega} - \bar{W}_1\Omega\big) + W_3 + J \wedge W_4 \;, \\ \mathrm{d}\Omega &= W_1 J \wedge J + J \wedge W_2 + \Omega \wedge W_5 \;, \end{aligned} \tag{I.38}$$

with $J \wedge J \wedge W_2 = J \wedge W_3 = \Omega \wedge W_3 = 0$. These equations can be inverted. As an example, the scalar W_1 is given by

$$\mathrm{d}\Omega \wedge J = \Omega \wedge \mathrm{d}J = W_1\, J \wedge J \wedge J \;, \quad W_1 \in \mathbb{C} \;. \tag{I.39}$$

We list a number of special cases

- $T^0 = 0$: Calabi-Yau; torsion-free $SU(3)$ structure: $Hol\big(\nabla^{L.C.}\big) = SU(3)$.
- $T^0 \in \mathcal{W}_1$: nearly Kähler
- $T^0 \in \mathcal{W}_2$: almost Kähler
- $T^0 \in \mathcal{W}_5$: Kähler; $U(3)$ structure.
- $T^0 \in \mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$: half-flat. These manifolds can be characterised by $\mathrm{d}\Omega_- = 0$ and $\mathrm{d}(J \wedge J) = 0$ [31, 32, 25].

E.2 G_2 structure

Now let M be a Riemannian 7-manifold M with one nowhere vanishing spinor θ. [3] Such seven-dimensional manifolds occur as background of M theory [33, 29]. In order to show that the existence of this spinor determines a G_2 structure on M, we compute the differential forms $\Omega^{(p)}$. By symmetry properties of spinor bilinears most of these form vanish:

p	0	1	2	3	4	5	6	7
$\Omega^{(p)}$	1	0	0	φ	$\star\varphi$	0	0	$\star 1$

We observe that the three-form

$$\varphi := \tfrac{1}{3!}\left(\bar{\theta}\gamma_{abc}\theta\right)\, e^a \wedge e^b \wedge e^c \tag{I.40}$$

fulfils the following equation, *cf.* (I.19):

$$\varphi_{abc}\varphi^{cde} = 2\delta_a^{[c}\delta_b^{d]} + (\star\varphi)_{ab}{}^{de}\ . \tag{I.41}$$

which can be shown by Fierz reordering [29]. Hence, φ determines a G_2 structure on M.

The intrinsic torsion of a G_2 structure is an element of

$$T^0\Big|_x \in \mathfrak{g}_2^{\perp} \otimes T_x^*\ . \tag{I.42}$$

[3]The supersymmetry parameter η is an anti-commuting spinor. In dimensional reduction, η is decomposed as $\eta = \epsilon \otimes \theta$ into an anti-commuting four-dimensional spinor ϵ and into θ, which then has to be a *commuting* spinor.

Similar to $SU(3)$ structures, the intrinsic torsion decomposes into four $\mathfrak{g}_2$ modules

$$T^0 \in \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \,, \tag{I.43}$$

which have dimension 1,14,27, and 7 [31]. The components $X_i \in \mathcal{X}_i$ are given by [28]

$$\begin{aligned} \mathrm{d}\varphi &= X_1 \star \varphi + X_4 \wedge \varphi + X_3 \\ \mathrm{d} \star \varphi &= \tfrac{4}{3} X_4 \wedge \star\varphi + X_2 \wedge \varphi \end{aligned} \tag{I.44}$$

A G_2 structure is called parallel, if $T^0 = 0$, or, equivalently, if it is calibrated, $\mathrm{d}\varphi = 0$ and co-calibrated, $\mathrm{d}\star\varphi = 0$. A parallel G_2 structure is also equivalent to $Hol\big(\nabla^{L.C.}\big) \subset G_2$ [27].

If there is more than one nowhere vanishing spinor on M, more differential forms can be built. This leads to a further reduction of the structure group of TM to proper subgroups of G_2, *i.e.* $SU(2) \subset SU(3) \subset G_2$. For instance, if there are two spinors θ^1 and θ^2 one can have a one-from

$$v := \left(\bar{\theta}^1 \gamma_a \theta^2\right) e^a \,, \tag{I.45}$$

in addition to φ. It can be shown that the structure group is the maximal subgroup $SU(3)$ of G_2. Also the intrinsic torsion can be decomposed into $\mathfrak{su}(3)$-modules, see Refs. [29, 28]. The nowhere vanishing one-form (I.45) determines a fibration of the seven-dimensional G_2 manifold over an interval. In Chapter III, we will explain how $SU(3)$ and G_2 structures are related.

II Five-dimensional Domain Walls

Singularities appear quite generically in classical gravity [34]. Here, we analyse curvature singularities of a class of five-dimensional domain-wall solutions of gauged supergravity [19, 20].

In [22, 23] it has been shown that there is a mechanism which prohibits naked singularities of electric and magnetic BPS solutions of ungauged five-dimensional supergravity when embedded into M theory compactified on a Calabi-Yau three-fold (without G-flux). This mechanism might be called "the intervention of additional states." Before a curvature singularity develops the low-energy effective action breaks down, and additional light modes need to be taken into account. In M theory these states descend from branes wrapped on cycles. The mechanism which avoids curvature singularities is an interplay between internal, compact space and non-compact space-time, similar to the so-called enhançon mechanism [18].

While the “enhançon-mechanism” for domain walls and electric BPS solutions was observed in particular models [22, 23], we are able to prove model-independently that one always encounters new M theory physics, such as additional light states, when, or even before a naked space-time singularity occurs [CM03]. We show explicitly how the behaviour of the metric of the Kähler cone on the boundaries is related to the geometrical degeneration and the new physics occurring there. As a by-product, we obtain various nice relations between geometrical quantities of the Kähler cone and of space-time. Our arguments can be adapted to five-dimensional black-hole solutions [CM03] and to cosmological solutions of five-dimensional supergravity [CM04].

In Section A we analyse the occurrence of space-time curvature singularities where we show that indeed singular solutions are as generic as nonsingular solutions. The situation, however, can be improved on by embedding the five-dimensional supergravity into eleven-dimensional M theory. The five-dimensional gauged supergravity action describes compactifications of M theory on Calabi-Yau manifolds in the presence of background flux [19, 21]. We review this relation in Section B. The corresponding eleven-dimensional domain walls are known as Hořava-Witten theory [35]. In this setup, the five-dimensional scalar fields have the interpretation of Calabi-Yau moduli. These moduli are Kähler moduli and hence they measure sizes of cycles in the Calabi-Yau manifold. Since the moduli space has the form of a cone, it is called the Kähler cone. We explore this connection in Section C.

A BPS Domain-wall Solutions of Five-dimensional Gauged Supergravity

A.1 Review of Domain-wall Solutions

In this subsection, we review the domain-wall solution of a class of five-dimensional gauged supergravity theories [36, 37, 38]. This class of supergravity theories describes the bulk dynamics of Hořava-Witten theory compactified on a Calabi-Yau three-fold [19, 20, 21, 39].

The gauged supergravity actions we consider here contain a potential for the scalar fields, which is such that neither flat Minkowski space nor AdS_5 space is a solution. The most symmetric solutions are $1/2$-BPS solutions, invariant under four supercharges and under four-dimensional Lorentz transformations, only. These domain walls are of Hořava-Witten type (HW), since they do not interpolate between two fully supersymmetric vacua. In general, the solutions exist only for a finite interval in y. Note that there are other domain-wall solutions of gauged supergravity (with a different gauging) [37, 38], which do interpolate between two maximally supersymmetric vacua *cf.* Fig. II.1. An example of these "GST-type" solutions is given by a class of solutions, which approaches for $|y| \to \infty$ two different maximally supersymmetric AdS_5 spaces.

The bosonic fields are part of the following multiplets: Metric and graviphoton, $\{g_{\mu\nu}, A_\mu\}$, belong to the gravity multiplet. There are $N-1$ vector fields and scalars, $\{A_\mu^{\hat{\imath}},\ \phi^{\hat{\imath}}\}$, $\hat{\imath} \in 1 \dots N-1$ in vector multiplets. Furthermore, the theory contains the so-called universal hypermultiplet (UHM), $\{V,\ a,\ \xi,\ \bar{\xi}\}$, which consists of two real scalars and of one complex scalar. The theory might contain additional hypermultiplets which, however, do not play a role in the domain-wall solutions we consider. In Chapter III, we construct the four-dimensional cousins of this solution, where an arbitrary number of hypermultiplets can vary. By supersymmetry, the scalar fields of the hypermultiplets have to be maps from space-time into a quaternion-Kähler target manifold. For details, we refer to Chapter III.

The scalar fields $\phi^{\hat{\imath}}$ parametrise a degree-three hyper-surface in $\mathbb{R}^N$ [36, 40, 41]

$$\mathcal{V}(X) := \frac{1}{6}\, c_{ijk} X^i X^j X^k = 1\ , \qquad i,j,k \in 1 \dots N\ , \tag{II.1}$$

determined by the real, symmetric, and constant coefficients c_{ijk}. As the graviphoton A_μ and the vector multiplet gauge fields $A_\mu^{\hat{\imath}}$ combine into N vector fields A_μ^i, we combine the $N-1$ scalars $\phi^{\hat{\imath}}$ and V together, anticipating the structure we will obtain by dimensional reduction, and define

$$Y^i := V^{1/6} X^i\ . \tag{II.2}$$

The field a in the UHM is called axion, since the action is invariant under shifts $a \to a + c$. The domain walls we consider here are solutions to gauged five-dimensional supergravity, where this shift symmetry has been gauged. This implies that the axion becomes charged under a combination of the vector fields A_μ^i. Moreover, the gauging induces a potential for the moduli. As a consequence, the most symmetric solutions are $1/2$-BPS domain walls, which are invariant under 4 supercharges and under four-dimensional Lorentz transformations, only.

The five-dimensional line element of such a domain-wall solution is given by [19, 20]

$$\mathrm{d}s^2 = \exp\big[2U(y)\big]\Big\{-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\Big\} + \exp\big[8U(y)\big]\mathrm{d}y^2 \,, \tag{II.3}$$

in terms of a single function U, which only depends on the transversal coordinate y. This function is related to the scalar moduli by

$$\exp\big[6U(y)\big] = V(y) = \left(\frac{1}{6}\, c_{ijk}Y^i(y)Y^j(y)Y^k(y)\right)^2 \,. \tag{II.4}$$

The moduli $Y^i(y)$, in turn, are determined in terms of harmonic functions $H_i(y)$,

$$c_{ijk}Y^j(y)Y^k(y) = 2\,H_i(y) \,, \qquad H_i(y) = a_i y + b_i \,, \quad a_i, b_i \in \mathbb{R} \,. \tag{II.5}$$

Note that the domain-wall solution is completely fixed by a flow on the scalar manifold which is parameterised by the transverse coordinate y. The solution starts at $y = y_1$ at a particular point on the scalar manifold and evolves as determined by the equations (II.3)–(II.5) until it terminates at a different point at $y = y_2$. Since the five-dimensional theory does not have fully supersymmetric ground states, there is no fixed-point behaviour and we have to introduce boundaries at the positions y_1, y_2 by hand *cf.* Fig. II.1. The so-called generalised stabilisation equations (II.5) are an universal feature of both, domain-wall and black-hole solutions [42], and therefore the following analysis of (space-time) curvature singularities can be adapted for black-hole solutions [CM03].

A.2 Curvature Singularities of Domain-wall Solutions

Here, we investigate the occurrence of space-time curvature singularities. We start by calculating the Ricci scalar of the metric (II.3) and then analyse possible sources of divergences. The Ricci scalar is given by ($' = \frac{\mathrm{d}}{\mathrm{d}y}$)

$$R = 4 \exp\big[-8U\big]\,\big(3U'U' - 2U''\big)\;. \tag{II.6}$$

This expression can diverge (U is related to V by equation (II.4)):

1. if either $\exp\big[-U\big] \to \infty$ ($V \to 0$),
2. or if the first or second derivatives of U (or V) diverge.

Since the line element (II.3) depends only on the function U, all components of the Riemann tensor are polynomials in U' and U''. Hence, our analysis applies to all curvature invariants of the domain-wall metric.

Since case (i) has already been covered in the literature [43], it remains to analyse the somewhat less obvious case (ii), *i.e.*, diverging curvature invariants at finite and non-zero V. It is convenient to consider the first derivative of $\sqrt{V}$ instead of V:

$$\left(\sqrt{V}\right)' = \frac{1}{2}\,c_{ijk}Y^iY^jY'^k = \frac{1}{2}\,Y^iH_i'\;, \tag{II.7}$$

where in the last step we have used the relation

$$c_{ijk}Y^jY'^k = H_i'\;, \tag{II.8}$$

which follows from differentiating (II.5) with respect to y. Since the harmonic functions H_i are at most linear in y, V' is regular as long as the moduli Y^i are finite [44, 22].

Differentiating equation (II.7) once more we find

$$\left(\sqrt{V}\right)'' = \frac{1}{2}\, Y'^i H_i' + \frac{1}{2}\, Y^i H_i'' \ . \tag{II.9}$$

Clearly, $(\sqrt{V})''$ can blow up if Y'^i diverges. By introducing the matrix [39]

$$\tilde{M}_{ij} = \frac{1}{2}\, c_{ijk} Y^k \ , \tag{II.10}$$

we can invert equation (II.8):

$$Y'^i = \frac{1}{2}\, \tilde{M}^{ik} H_k' \ . \tag{II.11}$$

Of course this inversion is only formal, because $\tilde{M}^{ij}$ depends on the moduli Y^i. Since $H_i' = a_i =$ const, $|Y'^i| \to \infty$ when $\tilde{M}_{ij}$ is not invertible, or, equivalently, when $\det \tilde{M} = 0$. Using the last equation, we obtain

$$\left(\sqrt{V}\right)'' = \frac{1}{4}\, H_i' \tilde{M}^{ij} H_j' + \frac{1}{2}\, Y^i H_i'' \ . \tag{II.12}$$

The appearance of the matrix $\tilde{M}_{ij}$ is the link between space-time curvature singularities and properties of the moduli-space metric, which we will deal with in Section C.

We have shown that there are two possible causes for curvature singularities of domain-wall solutions: (i) $V \to 0$, and (ii) $\tilde{M}_{ij}$ non-invertible at finite $V \neq 0$. Let us demonstrate that these curvature singularities do occur *generically*: Since V is a homogeneous function of degree three in the moduli Y^i, it will have zeros if the moduli Y^i are allowed to take arbitrary real values. This covers case (i). As for case (ii), a generic matrix $\tilde{M}_{ij}$ is invertible, but becomes singular in co-dimension one in parameter space. In other words, if no additional conditions on the parameters are imposed, the set of solutions will decompose into two subsets: those, which do not cross the hyper-planes where $\tilde{M}_{ij}$ becomes singular, and those which do. This can be seen explicitly in the examples considered in Ref. [22]. In both cases singular and non-singular space-time geometries are equally generic, and supergravity does not provide any constraints on the parameters which exclude the singular solutions. The difference between case (i) and case (ii) is that in the first case the metric on the scalar manifold diverges, while it develops a zero eigenvalue in the second case. In both cases the five-dimensional supergravity lagrangian becomes singular, which indicates that we need input from an underlying fundamental theory. We will see that this input is provided by M theory, if the supergravity theory is obtained as a Calabi-Yau compactification.

B Compactification of Eleven-dimensional Supergravity on Calabi-Yau Three-folds with Background Flux

Here, we recall how five-dimensional gauged supergravity can be obtained by compactification of eleven-dimensional supergravity on Calabi-Yau three-folds in the presence of background flux.

There are two points of view concerning the relation of gauged five-dimensional supergravity actions to eleven-dimensional supergravity theory: (i) compactification on a Calabi-Yau manifold, assuming that the flux only excites Calabi-Yau zero-modes and does not deform the Calabi-Yau structure. The presence of background flux is taken into account by including it as a "non-zero mode," see Refs. [19, 20, 21]. Or, (ii), compactification on a "deformed" Calabi-Yau manifold [19, 20, 45, 25].

We describe the first approach. The second approach is used in Chapter III. The bosonic fields of eleven-dimensional supergravity consist of the metric and of a three-form gauge potential C_3 with associated four-form field strength $G_4 = \mathrm{d}C_3$. We start by specifying a basis of the second cohomology group consisting of $h^{1,1}$ harmonic $(1,1)$ forms ω_i. The Kähler form can be expanded in this basis,

$$J = v^i \omega_i \ , \qquad \langle \omega_i \rangle = H^{1,1}(X) \ , \quad i = 1 \dots h^{1,1} := \dim H^{1,1}(X) \ , \qquad \text{(II.13)}$$

with real moduli v^i, which are related to the moduli of Section A by the rescaling $v^i := V^{1/3} X^i = V^{1/6} Y^i$. Since we will need a basis of $H^{2,2}(X)$ and of the even homology of X, we introduce dual 4-forms, 2-cycles, and 4-cycles. By Poincaré duality, there is a dual basis of 4-forms ν^i defined as

$$\int_X \nu^i \wedge \omega_j = \delta^i_j \ , \qquad \langle \nu^i \rangle = H^{2,2}(X) \ . \qquad \text{(II.14)}$$

In homology, we fix a basis of 2- and 4-cycles, with relations

$$\int_{C^i} \omega_j = \int_{D_j} \nu^i = \delta^i_j \ , \quad \langle C^i \rangle = H_2(X) \ , \quad \langle D_i \rangle = H_4(X) \ . \qquad \text{(II.15)}$$

The symmetric tensor c_{ijk} of Section A acquires now the interpretation of triple-intersection numbers

$$c_{ijk} = D_i \circ D_j \circ D_k = \int_X \omega_i \wedge \omega_j \wedge \omega_k \ , \qquad \text{(II.16)}$$

which implies that it is *integer valued*,[1] in contrast to pure five-dimensional supergravity where *real-valued* tensors are allowed.

Having introduced a basis for the even (co-)homology, we now describe how the bosonic fields of Section A descend from the fields of eleven-dimensional supergravity:

$$C_{MNP} \implies A^i_\mu \mathrm{d}x^\mu \wedge \omega_i \ , \quad \xi \Omega_{abc} \ , \quad \bar{\xi}\bar{\Omega}_{\bar{a}\bar{b}\bar{c}} \qquad \text{(II.17)}$$

$$G_{MNPQ} = (\mathrm{d}C)_{MNPQ} \implies \mathrm{d}a = \star_5 G \qquad \text{(II.18)}$$

$$g_{MN} \implies v^i \omega_i \ , \quad g_{\mu\nu} \qquad \text{(II.19)}$$

Here, Ω_{abc} denotes the holomorphic $(3,0)$ form which exists on every Calabi-Yau three-fold. The fields $X^i = V^{-1/3} v^i$ of Section A parameterise the relative sizes of the cycles of X, since they obey the constraint (II.1), whereas the UHM scalar V parameterises the volume of X. The axion a comes from dualising the dimensionally reduced 4-form field strength, and therefore has a shift symmetry: $a \to a + c$.

In general, dimensional reduction on a generic Calabi-Yau manifold yields more hypermultiplets than the UHM alone. For the type of domain-wall solutions we consider, these extra hypermultiplets are spectators, and it is a consistent truncation to keep these fields constant.

[1]This holds in an appropriate basis of (co-)homology and for non-singular X. For singular Calabi-Yau three-folds these numbers can be rational.

Following [19, 20, 21] we assume that the back-reaction of the flux on the geometry is such that it excites Calabi-Yau zero-modes only, and does not distort the Calabi-Yau structure. Since the background four-form flux is an element of $H^4(X) = H^{2,2}(X)$, it can be expanded as follows

$$G = \alpha_i \nu^i \in H^{2,2}(X) \ , \qquad \text{(II.20)}$$

with constants α_i subject to a quantisation condition [46]. However, within the supergravity approximation the flux parameters can be taken to be continuous as discussed in Ref. [25]. In the dimensionally reduced five-dimensional theory, turning on flux (II.20) leads to (i) a potential for the moduli v^i

$$\frac{1}{4V^2}\,\alpha_i \alpha_j G^{ij} \ , \qquad \text{(II.21)}$$

with G^{ij} the inverse metric on the moduli space, and (ii) to a gauging of the shift symmetry of the axion,

$$D_\mu a = \partial_\mu a + \alpha_i A^i_\mu \ . \qquad \text{(II.22)}$$

It is important to keep in mind that the domain-wall solutions of the last subsections are exact solutions of five-dimensional gauged supergravity theory, but do not lift to exact solutions of the eleven dimensional theory. The corresponding eleven-dimensional domain-wall solutions of Hořava-Witten theory are only known up to first order, and to this order they agree with the five-dimensional domain-wall solutions, see Ref. [20].

We have already mentioned that the tensor c_{ijk} has to be integer valued in a Calabi-Yau compactification. Similarly, the scalar fields v^i are subject to certain constraints we will deal with in the next section.

C Properties of the Kähler Cone

Having described five-dimensional domain-wall solutions from the point of view of supergravity in the last section, we now investigate the interplay between space-time physics and properties of the Kähler moduli space.

C.1 The Kähler Cone of Calabi-Yau Three-folds

By Wirtinger's theorem, the Kähler form measures the volume of holomorphic curves, surfaces and the volume of the Calabi-Yau manifold X. For all holomorphic curves $C \subset X$ and surfaces $S \subset X$, the following inequalities define the Kähler cone $\mathcal{K}$ [47]:

$$\mathrm{Vol}(C) = \int_C J > 0 \, , \tag{II.23}$$

$$\mathrm{Vol}(S) = \frac{1}{2!}\int_S J \wedge J > 0 \, , \tag{II.24}$$

$$V := \mathrm{Vol}(X) = \frac{1}{3!}\int_X J \wedge J \wedge J = \frac{1}{6}\, c_{ijk} v^i v^j v^k = \mathcal{V}(v) > 0 \, . \tag{II.25}$$

Thus, the Kähler moduli space has the structure of a cone. The (closure of) the Kähler cone is the cone $\overline{NE}^1(X)$ of *nef* classes, which is dual to (the closure of) the Kleiman-Mori cone $\overline{NE}_1(X)$ of effective 2-cycles [48, 49]. The duality is given by the pairing $\mathrm{Pic}(X) \times H_2(X) \longrightarrow \mathbb{Z}$, which is $\int_C L$ for a curve C and $L \in \mathrm{Pic}(X) = H^{1,1} \cap H^2(X)$, where $\mathrm{Pic}(X)$ denotes the Picard group of X. If X is a Calabi-Yau three-fold, then the Kähler cone is *locally* polyhedral away from the so-called cubic cone $W := \left\{ v^i \in \mathbb{R} \mid V = 0 \right\}$ [48]. For *toric-projective* Calabi-Yau varieties the Kähler moduli space is a strongly convex finite polyhedral cone [50, 51], and there is an explicit, global parameterisation, which takes the form

$$\mathcal{K} := \left\{ v^i \in \mathbb{R} \;\middle|\; 0 < v^i < \infty \,,\; 1 \le i \le h^{1,1} \right\} . \tag{II.26}$$

We call this parameterisation adapted, since the moduli v^i measure volumes of holomorphic 2-cycles C_i.

The metric on the Calabi-Yau Kähler moduli space is given by [52, 53]

$$G_{ij} := \frac{1}{2V} \int_X \omega_i \wedge \star\omega_j = -\frac{1}{2} \frac{\partial}{\partial v^i} \frac{\partial}{\partial v^j} \log \mathcal{V}(v) \,. \tag{II.27}$$

This metric is non-degenerate inside the Kähler cone. With the use of equation (II.25) it can be rewritten as

$$G_{ij} = -\frac{1}{V}\, M_{ik} \left(\delta^k_j - \frac{3}{2}\, T^k_j \right) \,, \qquad T^k_j := \frac{1}{6V}\, c_{jmn} v^m v^n v^k \,. \tag{II.28}$$

Here, $M_{ij} = V^{1/6} \tilde{M}_{ij}$ is a rescaled version of the matrix $\tilde{M}_{ij}$ introduced in (II.10). The matrix T is a projector, $T^2 = T$, of trace one. By the Hodge index theorem, the signature of the matrix

$$M_{ij} = \frac{1}{2} \int_X J \wedge \omega_i \wedge \omega_j = \frac{1}{2}\, c_{ijk} v^k \tag{II.29}$$

is $(1, h^{1,1} - 1)$ [48]. Since non-invertability of the matrix M is one cause of space-time curvature singularities (see Section A), equation (II.28) establishes the link between the occurrence of curvature singularities and properties of the Kähler-cone metric. This connection can be made more explicit by calculating the determinant of G:

$$\det G = \left(\frac{-1}{V} \right)^{h^{1,1}} \det M \; \det \left(\mathbb{1} - \frac{3}{2}\, T \right) = -\frac{1}{2} \left(\frac{-1}{V} \right)^{h^{1,1}} \det M \,, \tag{II.30}$$

where in the last step we have made use of the fact that T is a projector of trace one. There is a basis in which T assumes the form $T = \mathrm{diag}(1, 0, \dots 0)$, and we obtain

$$\det\left(\mathbb{1} - \frac{3}{2}\,T\right) = \det\mathrm{diag}\big(-1/2, 1, \dots, 1\big) = -1/2\ , \tag{II.31}$$

which completes the derivation of equation (II.30).

It is the aim of the next subsection to use the relation (II.30) in order to analyse regularity properties of the metric (II.27).

C.2 Degenerations of the Kähler-cone Metric and Singularities of Space-time

In this subsection, we analyse how the Kähler-cone metric (II.27) behaves on boundaries of the Kähler cone, in particular whether it develops zero eigenvalues. By "Kähler-cone metric at the boundary" we always mean the limit of the Kähler-cone metric as one approaches the boundary, and not the scalar metric of the extended effective field theories which explicitly include the additional light modes [43, 54, 55].

We consider boundaries of the Kähler cone where one particular 2-cycle, $C^\star$, collapses:

$$\partial_\star\mathcal{K} := \Big\{\big(v^{\tilde{\imath}} \neq 0,\ v^\star = 0\big)\ ,\quad 0 < \mathcal{V}(v) < \infty\Big\}\ ,\quad \tilde{\imath} \neq \star\ . \tag{II.32}$$

The contractions at these co-dimension-one faces are called primitive. In Calabi-Yau three-folds the following contractions can take place [48, 43]:

- Type I ("$2 \to 0$"): A finite number of isolated curves in the homology class $C^\star$ is blown down to a set of points, $\mathrm{Vol}(C^\star) = v^\star \to 0$, *e.g.*, with local geometry the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1$. [2]

- Type II ("$4 \to 0$"): A divisor $D = u^i D_i$ collapses to a set of points: $\mathrm{Vol}(D) \propto (v^\star)^2$.

- Type III ("$4 \to 2$"): A (complex) one-dimensional family of curves sweeps out a divisor $D = u^i D_i$. Contracting this family of curves induces a collapse of D to a curve of genus g: $\mathrm{Vol}(D) \propto v^\star$, *e.g.*, ($g = 0$ case) $\mathcal{O}(0) \oplus \mathcal{O}(-2) \longrightarrow \mathbb{P}^1$.

- Cubic cone ("$6 \to 4$", "$6 \to 2$", "$6 \to 0$"): These contractions correspond to $V \propto v^\star$, $V \propto (v^\star)^2$ and $V \propto (v^\star)^3$.

Note that our definition of boundaries $\partial_\star\mathcal{K}$ in equation (II.32) does not include the cubic cone, since this situation ($V \to 0$) is already covered by case (i) in Section A: the space-time Ricci scalar ($R \propto V^{-8/3}$) diverges when $V \to 0$.

Boundaries of type I and type III can be crossed into the Kähler cone of a new Calabi-Yau threefold, which is bi-rationally (and, for type III, even biholomorphically) equivalent to the original one. Crossing these boundaries corresponds to a flop [13] or going through gauge symmetry enhancement

[2] Here, $\mathcal{O}(-1)$ is the standard line bundle with Chern number -1 [56]. By the adjunction formula the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1$ can be shown to have a trivial canonical bundle, using that the Euler number of $\mathbb{P}^1$ is 2. Therefore, it can locally approximate a Calabi-Yau manifold.

[48, 16], respectively. The extended Kähler cone is gotten by enlarging the Kähler moduli space at all boundaries of type I. Enlarging in addition the Kähler moduli space at all boundaries of type III, one obtains the extended movable cone [13, 16]. However, this second extension only adds "gauge copies" to the parameter space (see for example [54, 23] for an explanation). While type-I and type-III boundaries are "internal boundaries" of the M theory moduli space, type-II contractions and the cubic cone lead to proper boundaries. At boundaries of type II the M theory moduli space ends, and it has been shown that the tension of strings descending from M5-branes wrapped on the divisor goes to zero at such boundaries[43]. Here the supergravity approximation breaks down, because infinitely many M theory states become massless. Similarly, the supergravity approximation breaks down when approaching the cubic cone, and in this case no interpretation in terms of M theory physics is known.[3]

Using equation (II.30), for finite and non-zero Calabi-Yau volume V, we are able to infer regularity properties of the Kähler-cone metric G from the matrix M and vice versa: there is a one-to-one map of zero eigenvalues of G to zero eigenvalues of M, *i.e.*, if

$$\det(M_{ij})\big|_{v^\star\to 0} \propto (v^\star)^n \ , \tag{II.33}$$

then there are n linearly independent eigenvectors of M (and of G) satisfying

$$u^i_{(a)} M_{ij}\big|_{v^\star=0} = 0 \ , \quad a = 1\ldots n \ . \tag{II.34}$$

[3]However, when dimensionally reducing on the M theory cycle, such regions correspond to non-geometrical phases of type-IIA string theory on the same Calabi-Yau manifold [43, 57].

Here and in the following, $|_{v^\star \to 0}$ denotes the limit approaching the boundary $\partial_\star \mathcal{K}$. Equation (II.34) is supposed to hold throughout the face $\partial_\star \mathcal{K}$. In particular, the null eigenvectors are determined by the triple intersection numbers, only. This implies that the components of the eigenvectors can be chosen to be *integer*. Hence, each zero eigenvector $u^i_{(a)}$ defines a divisor

$$D_{(a)} := u^i_{(a)} D_i \ . \qquad \text{(II.35)}$$

If there is a holomorphic surface within the homology class $D_{(a)}$, then its volume is given by

$$\begin{aligned} \frac{1}{2} u^i_{(a)} \int_X \omega_i \wedge J \wedge J &= \frac{1}{2}\, u^i_{(a)}\, c_{i\tilde{\jmath}\tilde{k}}\, v^{\tilde{\jmath}} v^{\tilde{k}} + u^i_{(a)}\, c_{i\tilde{\jmath}\star}\, v^{\tilde{\jmath}} v^\star + \frac{1}{2}\, u^i_{(a)}\, c_{i\star\star}\, v^\star v^\star \\ &= u^i_{(a)} M_{i\tilde{\jmath}}\big|_{v^\star=0}\, v^{\tilde{\jmath}} + 2\, u^i_{(a)} M_{i\star}\big|_{v^\star=0}\, v^\star + \frac{1}{2}\, u^i_{(a)}\, c_{i\star\star}\, v^\star v^\star \\ &= \frac{1}{2}\, u^i_{(a)}\, c_{i\star\star}\, v^\star v^\star \ , \end{aligned} \qquad \text{(II.36)}$$

where we have used equation (II.34). As a consequence, the divisors $D_{(a)}$, which are associated to null eigenvectors $u_{(a)}$, can never perform a type-III contraction, which is characterized by $\mathrm{Vol}(D) \propto v^*$. Irrespective of whether there exists a holomorphic surface in the class $D_{(a)}$, we learn that the moduli-space metric is always regular at boundaries of type I and type III. On the other hand, by definition of a type-II boundary ("$4 \to 0$", *i.e.* $\mathrm{Vol}(D) \propto (v^\star)^2$), we know that there exists at least one surface with homology class $D = u^i D_i$, which collapses to a point at $\partial_\star \mathcal{K}$. Hence the moduli-space metric develops a zero eigenvalue at boundaries of type II. At the cubic cone, the determinant of the moduli space generically diverges. More precisely, there are two cases: If $V \propto (v^\star)^3$, or $V \propto (v^\star)^2$, then the determinant of G always diverges, see equation (II.30), since the determinant

type of boundary	behaviour of metric
type I	$\det(G_{ij})\|_{\partial_\star\mathcal{K}} \neq 0$
type II	$\det(G_{ij})\|_{\partial_\star\mathcal{K}} = 0$
type III	$\det(G_{ij})\|_{\partial_\star\mathcal{K}} \neq 0$
cubic cone	divergent

Table II.1: Behaviour of Kähler moduli-space metric at boundaries of the Kähler cone.

of the matrix M can never compensate the zero in the denominator. What happens in the remaining case, $V \propto v^\star$ is that *generically* the determinant of G blows up, while at special points $\det M$ can compensate the zero in the denominator of equation (II.30). Table II.1 summarizes our result.

The generalisation of the proof to non-toric Calabi-Yau manifolds, where the global parameterisation of the Kähler moduli space which we have used above need not exist, is as follows. As in the proof above, at the cubic cone W the volume of X vanishes (*i.e.*., $V = 0$) and generically space-time curvature singularities occur in domain wall solutions. Moreover, the moduli-space metric diverges at the cubic cone. Since the Kähler cone is locally polyhedral away from W, we know that for each of the primitive faces there exists a local parameterisation of the form (II.32). There can be accumulation points of faces, but these are known to reside inside the cubic cone [48]. Thus, the proof is valid for *all* Calabi-Yau three-folds.

Now we are able to interpret the singularities of Section A in terms of M theory physics. Singularities of type (i) correspond to the cubic cone where the volume of the Calabi-Yau three-fold goes to zero. Singularities of type (ii) can occur in two different situations: The first is that one has reached a type-II boundary (*i.e.*, $\det G = 0$). On these boundaries the internal manifold and the effective supergravity lagrangian become singular, and tensionless strings appear, as discussed above. However, there is also the possibility that a singularity of type (ii) arises because one has crossed a boundary of type I or type III before, so that one is outside the Kähler cone. This situation is analogous to the enhançon mechanism [18]. When reaching boundaries of type I or type III, the triple-intersection numbers and therefore the low-energy equations of motion and the space-time metric change. Continuation of domain-wall solutions through type-I boundaries have been considered in Ref. [39], whereas continuation of black-hole and black-string solutions through type-I and type-III boundaries have been studied in Ref. [23]. Here we only need to use that type-I and type-III boundaries are internal boundaries of the extended Kähler cone, and that the metric of the extended Kähler cone does not become singular. After crossing such boundaries the moduli take values in another Kähler cone, and there our proof of absence of naked singularities applies again. In conclusion we see that in M theory singularities only occur on the boundary of the extended Kähler cone, where the internal manifold and the five-dimensional effective lagrangian become singular, and the description in terms of five-dimensional supergravity breaks down.

C.3 Example: The $\mathcal{F}_1$-Model

Here, we present an example of a Calabi-Yau manifold with $h^{1,1} = 3$ [58, 59], which has all the features discussed in the last subsections. It is an elliptic fibration over the first Hirzebruch surface $\mathcal{F}_1$. It turns out to be convenient to choose the following non-adapted parametrisation of the Kähler cone:[4]

$$\mathcal{K} = \left\{ S, T, U \in \mathbb{R} \,\middle|\, T > U > 0 \; , \quad S > \frac{T+U}{2} \right\} \; , \quad V = STU + \frac{1}{3}U^3 \; . \tag{II.37}$$

The matrices M_{ij} and G_{ij} take the form

$$M = \frac{1}{2}\begin{pmatrix} 2U & S & T \\ S & 0 & U \\ T & U & 0 \end{pmatrix} \; , \qquad G = \frac{1}{6V^2}\begin{pmatrix} U^4 + 3T^2S^2 & 2SU^3 & 2TU^3 \\ 2SU^3 & 3U^2S^2 & -U^4 \\ 2TU^3 & -U^4 & 3U^2T^2 \end{pmatrix} \tag{II.38}$$

with determinants

$$\det M = \frac{U}{4}\left(ST - U^2\right) \; , \qquad \det G = \frac{\det M}{2V^3} \; , \tag{II.39}$$

satisfying equation (II.30). Figure II.2 displays the Kähler cone of this model and table II.2 summarises the information in figure II.2 [59, 54, 23]. Note that the curves with $\det G = \infty$ ($U = 0$) and with $\det G = 0$ ($ST = U^2$), lie always outside or at boundaries of $\mathcal{K}$, in accord with the general statement of table II.1. The line $ST = U^2$ is called discriminant line, because $ST - U^2$ is the discriminant of a specific polynomial, whose zeros are in one-to-one correspondence with diverging derivatives of the scalar fields [23]. As long

[4] In this subsection, U denotes one of the scalar fields, and not the function appearing in the space-time metric.

location	name	physics	$\det(G_{ij})$
$U = 0$	cubic cone	unknown	divergent
$T = U$	type I	flop transition	regular
$S = \frac{T+U}{2}$	type III	$SU(2)$ symmetry enhancement	regular
$ST = U^2$	discriminant line	unphysical	degenerate

Table II.2: $\mathcal{F}_1$-model Kähler cone, *cf.* figure II.2

as $ST > U^2$, this polynomial does not have real zeros, and derivatives of scalar fields cannot diverge. Observe that the discriminant line lies beyond the type-III boundary, where gauge symmetry is enhanced. Therefore the naked space-time singularities occurring at $ST = U^2$ are unphysical [22, 23]. For the analogous black hole solution, the correct non-singular continuation beyond the type-III boundary is described in [23].

The extended Kähler cone of the model is obtained by extending it along the flop line, $S = (T + U)/2$. The flopped image of $\mathcal{K}$ has boundaries of type II, where the metric degenerates.

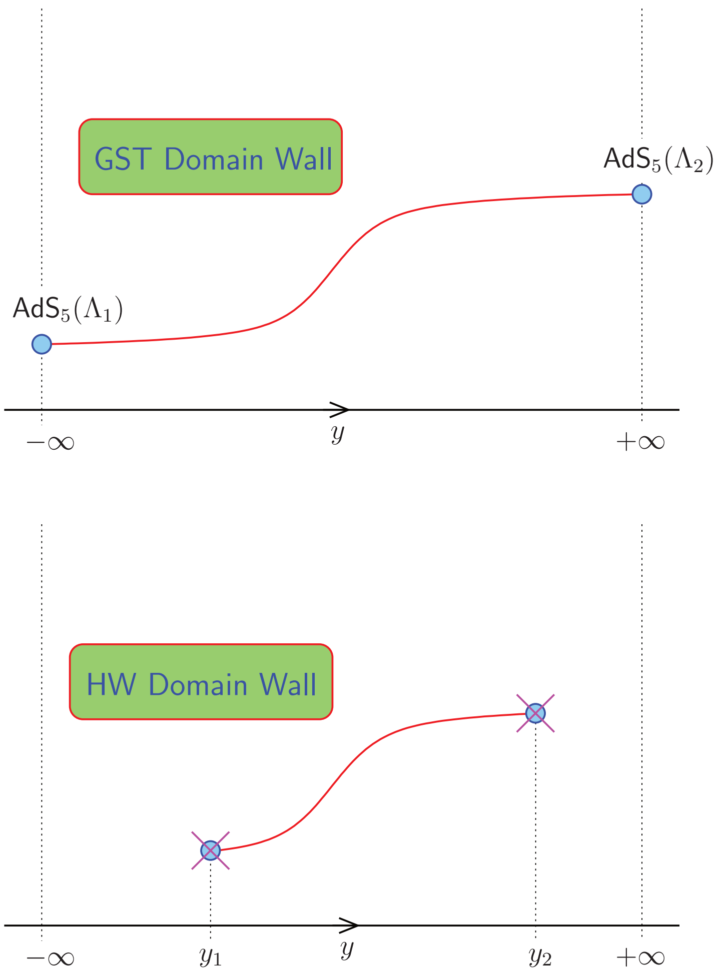
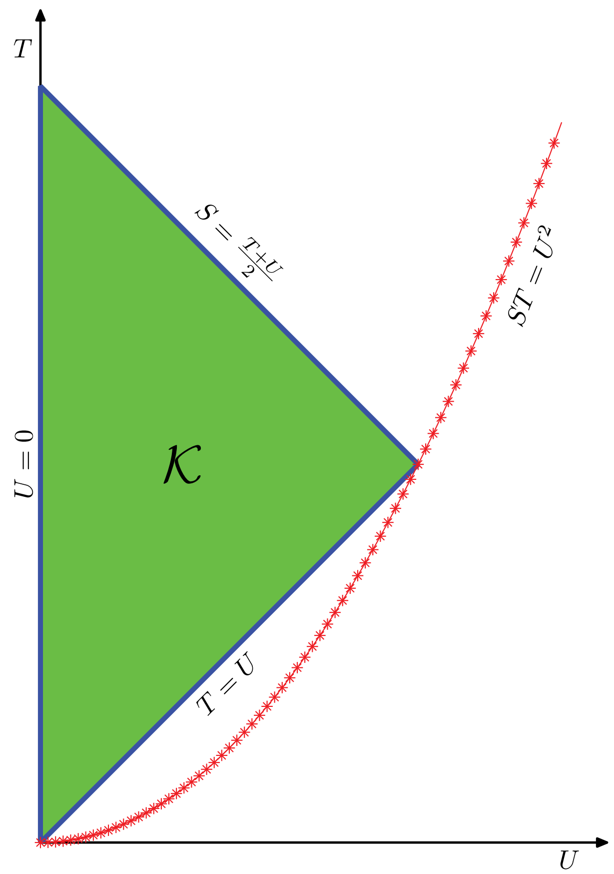


Figure II.1: GST *vs.* HW Domain Walls.



T
$S = \frac{T+U}{2}$
$ST = U^2$
$U = 0$
$\mathcal{K}$
$T = U$
U