# *Two-Dimensional Models of Black Hole Radiation*

Diploma thesis

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# Notation and Conventions

*General Conventions*

Wherever there are different conventions on signs in use in the literature (e.g. for the metric, the Riemann, and the Einstein tensor), we keep to the $(+ + +)$ convention in the terminology of [1].

This implies the Lorentzian metric signature $(- + \cdots +)$. We will need only spherical coordinates. For an event $x$ they are

$$\text{time: } x^t \equiv x^0, \text{ radius: } x^r \equiv x^1, \text{ elevation: } x^\theta \equiv x^2, \text{ and azimutal angle: } x^\varphi \equiv x^3.$$

In cases where there is no danger of confusion, e.g., if we consider only one spacetime point, we may write $t, r, \theta$, and $\varphi$ instead.

Wherever it is important to distinguish between quantities defined in spacetimes of different dimension, we indicate the dimension by a superscript in parentheses, e.g.

$$g^{(2)}_{\mu\nu} \, , \quad \sqrt{-g^{(3)}} \, , \quad R^{(4)} \, .$$

Variations and differential operators always act on all the terms written to the right of them. If there are brackets around the expression including the operator, then its action is restricted to inside the brackets. See, as an example, eqn. (6.15).

Units are chosen such that $c = G = \hbar = k_\mathrm{B} = 1$. Here, $c$ is the speed of light, $G$ denotes Newton's gravitational constant, $\hbar$ is Planck's constant, and $k_\mathrm{B}$ denotes Boltzmann's constant.

*Definitions*

The spatial angular element on the two-dimensional unit sphere is given by

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2 \qquad \text{where} \qquad \theta \in [0, \pi] \text{ and } \varphi \in [0, 2\pi) \, . \tag{1.1}$$

In spherical coordinates, spacetime intervals are determined by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \, .$$

They are classified as follows:

$$\mathrm{d}s^2 \begin{cases} < 0 & \text{for timelike} \\ = 0 & \text{for lightlike (null)} \\ > 0 & \text{for spacelike} \end{cases} \quad \text{intervals.} \tag{1.2}$$

In the context of two-dimensional Minkowski space, the finite interval between two events $x$ and $y$ is usually defined as

$$\sigma_{xy} = \frac{1}{2}\big((x^r - y^r)^2 - (x^t - y^t)^2\big) = \frac{1}{2}(x^u - y^u)(x^v - y^v) \ . \tag{1.3}$$

Symmetrization of two indices is denoted by parentheses,

$$A^{(\mu}B^{\nu)} = \frac{1}{2}(A^\mu B^\nu + A^\nu B^\mu) \ , \tag{1.4}$$

and antisymmetrization by square brackets,

$$A^{[\mu}B^{\nu]} = \frac{1}{2}(A^\mu B^\nu - A^\nu B^\mu) \ . \tag{1.5}$$

As (anti-)symmetrization involving more than two indices doesn't occur in the course of this work, it is implicitly understood that any indices occurring between those immediately adjacent to parentheses or square brackets are not affected by (anti-)symmetrization. In the following example, this applies to $\alpha$:

$$A_{(\mu\alpha\nu)} = \frac{1}{2}(A_{\mu\alpha\nu} + A_{\nu\alpha\mu}) \ , \qquad A_{[\mu}B_{\alpha\nu]} = \frac{1}{2}(A_\mu B_{\alpha\nu} - A_\nu B_{\alpha\mu}) \ .$$

The quantities of Riemannian geometry – Christoffel symbols, the Riemann and Ricci tensors, and the curvature (Ricci) scalar – are then defined (in the order given)[1]:

$$\begin{aligned}
\Gamma^\alpha{}_{\beta\gamma} &= \frac{1}{2}g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) \ , && (1.6)\\
R^\mu{}_{\nu\alpha\beta} &= \partial_{[\alpha}\Gamma^\mu{}_{\nu\beta]} + \Gamma^\mu{}_{\sigma[\alpha}\Gamma^\sigma{}_{\nu\beta]} \ , && (1.7)\\
R_{\mu\nu} &= R^\alpha{}_{\mu\alpha\nu} \ , && (1.8)\\
R &= R^\mu{}_\mu \ . && (1.9)
\end{aligned}$$

The last relation is independent from the sign conventions. In addition, the Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \ . \tag{1.10}$$

*Functions and distributions*

For the sake of clarity, we give the definitions of Dirac's and Heaviside's functions as we will use them.

Dirac's $\delta$ function (which is actually a distribution or generalized function) in $n$ dimensions:

$$\int \mathrm{d}^2y\, \delta^n(y-x)\Psi(y) = \Psi(x) \ , \quad \delta^n(y-x) = \delta(y^1-x^1)\delta(y^2-x^2)\cdots\delta(y^n-x^n) \ . \tag{1.11}$$

In contrast, the covariant $\delta$ function:

$$\int \mathrm{d}^2y\, \sqrt{-g(y)}\, \delta^n(y,x)\Psi(y) = \Psi(x) \ , \quad \delta^n(y,x) = \frac{1}{\sqrt{-g(y)}}\delta^n(y-x) \ . \tag{1.12}$$

The step function (Heaviside's function):

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \ . \tag{1.13}$$

*Green functions*

We will describe solutions to differential equations in terms of Green functions.

Denote by $G^m(x,y)$ the general massive Green function of an operator $\hat{F}$ acting on the coordinates of an event $x$: If

$$\hat{F}G^m(x,y) = \hat{F}_x G^m(x,y) = \delta^n(x,y) \tag{1.14}$$

then the inverse wave operator is defined as follows:

$$\left[\frac{1}{\hat{F}}\Psi\right](x) = \left[\frac{1}{\hat{F}}_y \Psi(y)\right](x) = \int \mathrm{d}^2y\,\sqrt{-g}\,G^m(x,y)\Psi(y)\ . \tag{1.15}$$

In the massless case we write

$$G(x,y) \equiv G^m(x,y)\big|_{m=0}\ . \tag{1.16}$$

Finally, we derive a property of symmetric Green functions. If $G^m(x,y) = G^m(y,x)$, we have

$$\begin{aligned}\int \mathrm{d}^2x\,\sqrt{-g(x)}\,\Psi(x)&\left[\int \mathrm{d}^2y\,\sqrt{-g(y)}\,G^m(x,y)\Phi(y)\right]\\ &= \int \mathrm{d}^2y\,\sqrt{-g(y)}\left[\int \mathrm{d}^2x\,\sqrt{-g(x)}\,G^m(y,x)\Psi(x)\right]\Phi(y)\ .\end{aligned}$$

If we relabel the integration variables on the right-hand side, $x \leftrightarrow y$, we obtain

$$\int \mathrm{d}^2x\,\sqrt{-g}\,\Psi\frac{1}{\hat{F}}\Phi = \int \mathrm{d}^2x\,\sqrt{-g}\left[\frac{1}{\hat{F}}\Psi\right]\Phi\ . \tag{1.17}$$

*Variational formulae*

We will need the following variations of curvature terms and Green functions:

$$\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}\ , \quad \text{therefore} \quad A_{\mu\nu}\delta g^{\mu\nu} = -A^{\mu\nu}\delta g_{\mu\nu}\ , \tag{1.18}$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}\ , \tag{1.19}$$

$$\int \mathrm{d}^2x\,\sqrt{-g}\,(\delta R)S = \int \mathrm{d}^2x\,\sqrt{-g}\,(\delta g_{\mu\nu})(S^{;\mu\nu} - g^{\mu\nu}S^{;\varepsilon}{}_{;\varepsilon} - R^{\mu\nu}S)\ , \tag{1.20}$$

$$\int \mathrm{d}^2x\,\sqrt{-g}\,A(\delta\Box)B = \int \mathrm{d}^2x\,\sqrt{-g}\,\left(\delta g_{\mu\nu}\right)\left[A^{,(\mu}B^{,\nu)} - \frac{1}{2}g^{\mu\nu}(A^{,\varepsilon}B_{,\varepsilon} + AB^{;\varepsilon}{}_{;\varepsilon})\right]\ , \tag{1.21}$$

$$\delta\frac{1}{m^2-\Box} = \frac{1}{m^2-\Box}(\delta\Box)\frac{1}{m^2-\Box} \qquad \text{and} \qquad \delta\frac{1}{\Box} = -\frac{1}{\Box}(\delta\Box)\frac{1}{\Box}\ . \tag{1.22}$$

For relations (1.20) and (1.21), we refer to A. Zelnikov, for (1.22) see [2].

Chapter 1

# Introduction

This diploma thesis is concerned with the quantum field theoretical observation that Black Holes are not entirely black but instead radiate particles with a thermal energy distribution. This phenomenon has been known since the mid-seventies and is named, after its discoverer, the Hawking effect [3].

The present work consists of two main parts: First, a general discussion of Black Holes and the Hawking effect that basically reflects the process of acquainting myself with the subject. Here, the basic formulae needed in the later chapters are introduced. The second part lends itself to two-dimensional models of gravity, mostly dilaton models, and the problem of defining them in such a way that their results qualitatively agree with the four-dimensional theory.

The reader is assumed to be familiar with Einstein's theory of General Relativity (which predicts the objects of our primary interest: Black Holes) and the theory of quantum fields in flat space. However, chapter 2 gives a short account of Schwarzschild Black Holes as a solution to Einstein's equations, of the Schwarzschild geometry, and of Penrose diagrams, a means to visualize infinitely extended regions of spacetime in finite-sized diagrams.

Those aspects of quantum field theory in a general curved spacetime which are needed in this work will be developed along the way. Here, [4] will be the primary reference.

*Black Holes and the Hawking effect*

In 1916, Karl Schwarzschild found his famous solution to Einstein's equations for a mass distribution consisting of – and hence, a classical energy momentum tensor corresponding to – a point mass $M$ located at the origin of spatial coordinates but an otherwise perfectly empty space. This solution gives rise to spherical objects of radius $2M$ that have zero angular momentum and electric charge.

Such objects have been referred to as Black Holes since no physical entity – subject to the limit of the speed of light, $c$, as the maximum speed possible – can escape from their interior due to the mass' gravitational attraction. This holds for massive particles as well as for massless photons and, in fact, any kind of information.

All the more surprising was Hawking's announcement that from the quantum field theoretical point of view, 'Black Holes are not black'. This statement has to be understood in the sense that if the spacetime surrounding a Black Hole is supposed to be filled with classical vacuum, nevertheless a thermal flux of energy or thermal radiation of particles, resp., can be measured to come from it according to quantum field theory. The temperature of that radiation, called the Hawking temperature $T_{\mathrm{H}}$, has been calculated to be

$$T_{\mathrm{H}} = \frac{1}{8\pi M} \ .$$

The reason for this phenomenon is – as will be pointed out in chapter 4 – that the notion of vacuum is ambiguous in quantum field theory. We obtain Hawking radiation if we consider a quantum state which corresponds to vacuum defined in the region light rays going into the Black Hole come from, but measure the particle flux in the region they go to. In these regions, called past and future lightlike infinity (see chapter 2) or 'in' and 'out' region, particles and thus vacuum can be defined unambiguously, but the two definitions do not correspond to the same quantum state.

Furthermore, Hawking radiation is an important piece in the theory of the thermodynamics of Black Holes as it adds to the analogy between geometrical quantities appearing in the context of Black Holes and thermodynamical quantities the concept of temperature.

Chapter 3 will demonstrate how a non-Minkowskian spacetime can 'produce' particles by the mechanism just outlined. There, the simple example of a conformally flat spacetime expanding suddenly at $t = 0$ is considered. Although Hawking radiation is concerned with a metric depending on the radial coordinate instead of time, it is helpful to understand the mechanism of particle creation and – technically – Bogoliubov transformations explained by the simpler example in order to understand the nature of Hawking radiation.

Chapter 4 lends itself to the actual Hawking effect in a four-dimensional spacetime. Three approaches to the explanation of the effect will be given: an intuitive one considering pairs of virtual particles being torn apart by the Black Hole's gravitational field, an attempt to describe the scattering of field modes responsible for the different notions of particles in the 'in' and 'out' region by reducing the problem to potential scattering, and an outline of Hawking's derivation of Black Hole radiation.

*Two-dimensional dilaton models and the effective action approach*

Although the Hawking radiation itself can be calculated in quantum field theory, the same cannot be said for other effects. Already the computations needed to obtain the energy momentum tensor of Hawking radiation are very laborious and involved, making use of techniques like regularization and renormalization.

One interesting issue is the influence of the energy flux from the Black Hole on its geometry, known as the back-reaction. As the Black Hole radiates off energy and thereby loses mass, it decreases in size and in the end, evaporates. This process can, however, not yet be formulated in a mathematically consistent way, even in the semi-classical approximation.

People hope to learn about such mathematically complicated questions by considering substantially simplified models: they assume spherical symmetry of the geometry (given in the Schwarzschild case) and fields and, by restricting their attention to the time and radial coordinate only, reduce the problem to two dimensions.

Several such approaches to dimensional reduction will be reviewed in chapter 5. Two aspects have to be considered: how to dimensionally reduce the gravitational action, and how to do so with the matter action. Both times, the possibilities are either to ignore the spherical variables, or to integrate all quantities with respect to them. The latter procedure leads to the so-called dilaton gravity.

It will turn out that the dilaton model will be conceptually preferable to the inherently two-dimensional one as it is rooted in the four-dimensional theory and can thus be expected to yield results which are better comparable to those from the full four-dimensional theory.

As the trace of the quantum energy-momentum tensor of a field in its vacuum state is non-zero – which is known as the conformal trace anomaly in explained in chapter 5 – it can be used to compute an action which then plays the rôle of the matter action in the two-dimensional theory. It is called the effective action. It is required to lead to an energy-momentum tensor whose trace is just the trace anomaly, but it contains only geometrical quantities instead of the quantum fields.

It turns out, however, that the Hawking radiation obtained from this anomaly induced effective action in the dilaton model leads to disagreement with the four-dimensional theory as it implies a negative energy flux from the Black Hole.

The anomaly-induced effective action has been the subject of some debate. There has been disagreement on the coefficient of a particular term appearing in that action, and it was part of the task of this diploma thesis to investigate this ambiguity. In the course of the work I convinced myself that there is actually no ambiguity but rather a variety of different models considered, as had been published by J.S: Dowker [5] shortly before. During 1999 when I worked on my thesis, this explanation gained acceptance among the colleagues. The solution of the apparent ambiguity of the anomaly-induced effective action will be explained in chapter 5.

There have been attempts to restore a positive flux by adding conformally invariant terms to the effective action. These don't change the trace of the corresponding energy-momentum tensor (see chapter 5) but might contribute a useful correction to the individual components. Two such attempts will be described: the one suggested by V. Mukhanov, A. Wipf, and A. Zelnikov [6] which is not able to produce qualitative agreement between the two- and four-dimensional theories for all vacuum states, and the one proposed by R. Balbinot and A. Fabbri [7] which can do so but is obtained in a rather ad hoc way.

Chapter 6 is dedicated to a third proposal, made by Y. Gusev and A. Zelnikov [8] which is derived via heat kernel regularization. The implications of their conformally invariant contribution to the effective action have not yet been investigated. We will be able to compute the components of the energy-momentum tensor corresponding to this contribution, up to the numerical treatment of the integrals obtained. Once those integrals are known at least numerically, one can decide what the proposal by Gusev and Zelnikov implies for the Hawking flux in the dilaton model.

*Criticism of the model*

There are two issues about the two-dimensional models considered in this diploma thesis which are subject to criticism:

First, it is not clear whether two-dimensional models built on an effective action can reproduce the spherically symmetric four-dimensional theory. The trouble is not so much with the dimensional reduction; it is the fact that presently no procedure for a strict computation of the effective action exists that is proven to work. Whether the effective action approach is able to yield a satisfactory two-dimensional theory will probably become clear only when a full four-dimensional theory of quantum Black Holes exists.

Second, the models considered in this work and those referred to are all semiclassical approximations. Also their reliability can probably be judged only by comparison with a full theory of quantum gravity. In particular, it might turn out that Hawking radiation even prevents Black Holes from being created by gravitational collapse as they evaporate in the process. See also chapter 5 on this issue.

# Part I

# *Black Holes & the Hawking Effect in Four Dimensions*

Chapter 2

# The Spacetime of Black Holes

## 2.1 The exterior Schwarzschild solution

### 2.1.1 The Schwarzschild metric as a solution to Einstein's equations

*The solution to Einstein's equations for a point mass*

According to Einstein's theory of General Relativity, the effect of a gravitating mass on its surroundings can be expressed in purely geometrical terms. In the Newtonian theory, bodies move through ordinary space, their gravitational interaction causing them to deviate from trajectories which would be straight otherwise (given the absence of other forces). In Relativity, they always move along – properly defined – 'straight' trajectories, but through a spacetime 'curved' by their gravitating masses. That is, the flat and static metric of space is replaced by the metric of spacetime which depends on the distribution of matter (mass as well as energy). This dependence is determined by Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

where $R_{\mu\nu}$ and $R$ are defined by (1.8) and (1.9) and $T_{\mu\nu}$ is the matter stress energy tensor.

Consider a distribution of total mass $M$ concentrated on the origin of coordinates, implying spherical symmetry. Assuming zero electrical charge and angular momentum, the geometry away from the origin is the Schwarzschild geometry [1]

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - 2M/r} - r^2 \,\mathrm{d}\Omega^2 \tag{2.1}$$

where $\mathrm{d}\Omega^2$ is defined by (1.1).

*A few properties of Schwarzschild spacetime*

- The most prominent property of Schwarzschild spacetime is the fact that nothing that moves as fast as or even slower than light can escape the region inside the sphere of radius $r_S = 2M$. This radius is called the Schwarzschild radius. The surface of the sphere is referred to as the event horizon, and the interior region as a Black Hole, as no matter or information can propagate from inside the horizon outside.

- Schwarzschild coordinates have a singularity at $r = 2M$ whereas there is no physical singularity there; the only physical singularity is at $r = 0$. One can see the regularity of spacetime by considering an observer falling freely through the event horizon. He will not experience infinite forces or, more generally, measure quantities that become infinite just at the horizon [1]. Mathematically, the curvature scalar of the Schwarzschild metric does not diverge at the horizon.

- Although matter falling into the Black Hole arrives at the singularity within finite proper time (and light does so at a finite value of the affine parameter of its trajectory), the Schwarzschild time $t$ goes to infinity at the moment of crossing the horizon and decreases after [1]. Thus, an observer watching signals sent back from some object that falls into the Black Hole will never see it disappear. The object appears to approach the horizon more and more slowly, and the signals received become more and more red-shifted.

- The rôle of space and time inside the horizon is just the reverse of that outside in the following sense: An observer in the exterior region, in moving along his geodesic trajectory in the direction of growing proper time or – in the limit of motion at the speed of light – growing affine parameter, will inevitably cross every existing hyperplane of constant time exactly once on his way to future infinity but may stay at the same point in space.

  Such an observer is bound to cross every hypersurface of constant radius exactly once on his way into the singularity at $r = 0$ but may stay at a constant time coordinate. The only exception is the motion along the horizon.

  This property, stated here rather roughly and qualitatively, will become clearer when Kruskal diagrams are available for illustration.

*Curvature in two and four dimensions*

In four dimensions, the Ricci tensor $R_{\mu\nu}$ as well as the curvature scalar $R$ of the metric (2.1) vanish identically everywhere except at the origin. Outside the event horizon we have

$$R^{(4)} \equiv 0 \ , \qquad R^{(4)}_{\mu\nu} \equiv 0 \ .$$

For a two-dimensional Schwarzschild Black Hole the metric of which is just the $(t, r)$ part of the metric (2.1),

$$\mathrm{d}(s^{(2)})^2 = \left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - 2M/r} \ ,$$

these quantities behave differently. The two-dimensional curvature terms outside the event horizon at $r = 2M$ are

$$R^{(2)} = -\frac{4M}{r^3} \ , \qquad R^{(2)}_{tt} = \frac{2M(r-2M)}{r^4} \ , \quad R^{(2)}_{rr} = -\frac{2M}{r^2(r-2M)} \ , R^{(2)}_{tr} = R^{(2)}_{rt} \equiv 0 \ .$$

Even though they are not as simple as the four-dimensional pendants, these terms satisfy a simple relation which is, however, not specific to the Schwarzschild geometry but holds for any two-dimensional spacetime,

$$\frac{1}{2} R^{(2)} g^{(2)}_{\mu\nu} - R^{(2)}_{\mu\nu} \equiv 0 \ . \tag{2.2}$$

### 2.1.2 Kruskal coordinates

*A set of regular coordinates*

One possible set of coordinate transformations (with coefficients chosen according to [4]) removing the spurious singularity of Schwarzschild coordinates at the horizon reads

$$U = \mp 4M\, e^{-u/4M} \;, \qquad V = 4M\, e^{v/4M} \qquad \text{for } r \gtrless 2M \tag{2.3}$$

$$\text{where} \qquad u = t - r^* \;, \qquad v = t + r^* \tag{2.4}$$

$$\text{and} \qquad r^* = r + 2M \ln\left|\frac{r}{2M} - 1\right| \;. \tag{2.5}$$

As $r \to 2M \pm 0$, $r^* \to -\infty$ so $U \to 0$, thus the change of sign of $U$ doesn't cause any discontinuity.

$U$ and $V$ are called Kruskal coordinates. They have the following properties:

- Already the 'simple' null coordinates $u, v$ yield a conformally flat $(t, r^*)$-part of the line element:

  $$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right) \mathrm{d}u\, \mathrm{d}v - r^2\, \mathrm{d}\Omega^2 \;. \tag{2.6}$$

  In order to obtain (2.6),

  $$\mathrm{d}r^* = \frac{r}{r - 2M}\, \mathrm{d}r \qquad \text{for } r > 0 \tag{2.7}$$

  has been used.

- In Kruskal coordinates, the line element reads

  $$\mathrm{d}s^2 = \frac{2M}{r} e^{-r/2M}\, \mathrm{d}U\, \mathrm{d}V - r^2\, \mathrm{d}\Omega^2 \;.$$

  This metric has a conformally flat $(t, r)$ part as well, and the artificial (coordinate) singularity at $r = 2M$ has been removed.

*Analytic completion of the spacetime manifold*

- Kruskal coordinates give rise to a doubling of the spacetime manifold exterior of the horizon:

  The map $r \leftrightarrow r^*$ is bijective for $r > 2M$ and $r < 2M$ separately, and $U$ and $V$ can be uniquely determined from $r$ and $r^*$ and vice versa. Yet looking at the range of $U$ and $V$ as obtained by (2.3), one realizes that only half the $(U, V)$ plane is covered (see fig. 2.1).

  An attempt to calculate $r$ and $t$ from pairs $(U, V)$ taken from the other half yields perfectly well-behaved values; in fact, two pairs $(U, V)$ and $(-U, -V)$ correspond to each $(t, r)$:

  $$UV = \mp 16M^2\, e^{r^*/2M} \;, \quad \frac{U}{V} = -e^{t/2M} \qquad (\text{for } r \gtrless 2M) \;. \tag{2.8}$$

  So two more transformations can be introduced:

  $$U = \pm 4M\, e^{-u/4M} \;, \qquad V = -4M\, e^{v/4M} \qquad \text{for } r \gtrless 2M \;.$$

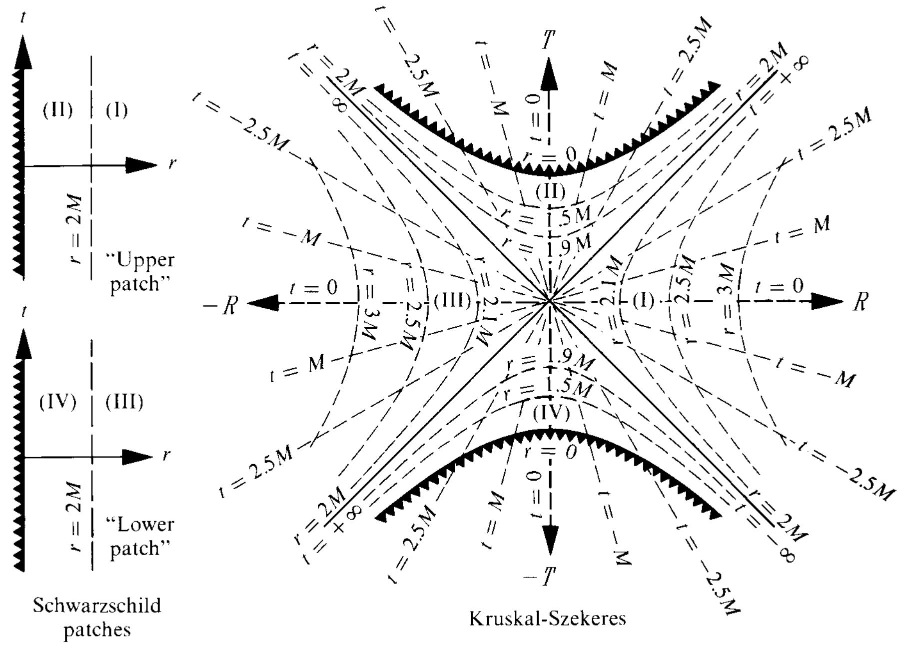


Figure 2.1: A Kruskal diagram which clearly shows the doubling of the space-time manifold by the introduction of Kruskal coordinates (taken from [1], fig. 31.3). Kruskal 'time' and 'radius' are defined in (2.9).

- Kruskal time and space coordinates can be defined from $U$ and $V$ by

$$T = \frac{1}{2}(U + V) , \qquad R = \frac{1}{2}(V - U) . \tag{2.9}$$

  Thus, two pairs $(T, R)$ exist for each point $(t, r)$.

  Table 2.1 lists the complete set of transformations needed to cover the whole plane of Kruskal time and radius.

- The transformation can be interpreted in the following way: A whole plane of Kruskal coordinates $U$ and $V$ is plotted. Each quadrant corresponds to one of the regions I - IV as a portrait of the whole or the left half of a $(t, r^*)$ plane, resp. The map applied is $x \to e^x$ acting on both null directions, mapping $(-\infty, \infty) \to [0, \infty)$. Since the plane is shrunk in the null directions, these directions are kept invariant while axes $t =$ const and $r^* =$ const are transformed into hyperbolae, particularly those representing the singularity, namely $r^* = 0$ in regions II and IV.

- Radial null rays make angles of 45 degrees with respect to $T$ and $R$. This is because $U$ and $V$ are obtained by purely stretching the $u$ and $v$ axes which are null geodesics and make angles of 45° with the $t$ and $r^*$ axes themselves. In contrast, null geodesics plotted in $(t, r)$ coordinates look like the plot of $r^*$ itself (fig. 2.2), up to shifting and mirroring. Let's check that $t = \pm r^* + c$ describes null geodesics:

  A light ray moving radially in- or outward will keep its angular coordinates because of the symmetries of the metric, thus we have $\mathrm{d}\Omega^2 = 0$. Hence $t = \pm r^* + c$ describes

Table 2.1: The complete set of Kruskal coordinate transformations. The regions I - IV are labeled according to fig. 2.1 and [1]. Null coordinates: $u = t - r^*$, $v = t + r^*$ (see eqn. (2.4)), abbreviations: $A = e^{r^*}\big|_{r>2M} = \sqrt{r/2M - 1}\, e^{r/4M}$, $B = e^{r^*}\big|_{r<2M} = \sqrt{1 - r/2M}\, e^{r/4M}$

| Region | $r$ | $U/4M$ | $V/4M$ | $T/4M$ | $R/4M$ |
|---|---|---|---|---|---|
| I | $> 2M$ | $-e^{-u/4M}$ | $e^{v/4M}$ | $A \sinh \frac{t}{4M}$ | $A \cosh \frac{t}{4M}$ |
| II | $< 2M$ | $e^{-u/4M}$ | $e^{v/4M}$ | $B \cosh \frac{t}{4M}$ | $B \sinh \frac{t}{4M}$ |
| III | $> 2M$ | $e^{-u/4M}$ | $-e^{v/4M}$ | $-A \sinh \frac{t}{4M}$ | $-A \cosh \frac{t}{4M}$ |
| IV | $< 2M$ | $-e^{-u/4M}$ | $-e^{v/4M}$ | $-B \cosh \frac{t}{4M}$ | $-B \sinh \frac{t}{4M}$ |

null rays,

$$
\begin{aligned}
\mathrm{d}s^2 &= \left(1 - \frac{2M}{r}\right)(\mathrm{d}r^*)^2 - \frac{\mathrm{d}r^2}{1 - 2M/r} \\
&= \left(1 - \frac{2M}{r}\right)\left(\mathrm{d}r + \frac{2M}{r/2M - 1}\,\frac{\mathrm{d}r}{2M}\right)^2 - \frac{\mathrm{d}r^2}{1 - 2M/r} = 0 \ ,
\end{aligned}
$$

and satisfies the geodesic equation

$$
\frac{\mathrm{d}x^\alpha}{\mathrm{d}r} + \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}x^{\alpha\,\beta}}{\mathrm{d}r}\,\frac{\mathrm{d}x^\gamma}{\mathrm{d}r} = 0 \tag{2.10}
$$

where the derivatives are taken, in the first place, with respect to some affine parameter of the curve which we then choose to be $r$ [9]. We only have to check the $t$ and $r$ components. We have

$$
\begin{aligned}
&\frac{\mathrm{d}r}{\mathrm{d}r} = 1 \ , \quad \frac{\mathrm{d}^2 r}{\mathrm{d}r^2} = 0 \ , \qquad \frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1}{1 - 2M/r} \ , \quad \frac{\mathrm{d}^2 t}{\mathrm{d}r^2} = \frac{-2M/r^2}{(1 - 2M/r)^2} \\
&\Gamma^r_{tt} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) \ , \quad \Gamma^t_{tt} = \frac{M}{r^2}\,\frac{1}{1 - 2M/r} \ , \quad \Gamma^r_{rr} = -\frac{M}{r^2}\,\frac{1}{1 - 2M/r} \ .
\end{aligned}
$$

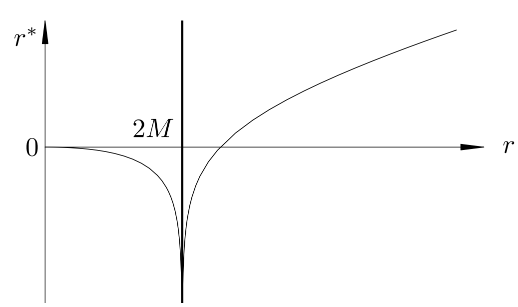


Figure 2.2: The 'tortoise' coordinate $r^*$ plotted against $r$ (see eqn. (2.5)). Radially in- and outgoing null geodesics $t = \pm r^* + c$ in the Schwarzschild geometry look like this. The event horizon is the vertical line $r = 2M$.

All other non-angular Christoffel symbols vanish. With these, (2.10) is satisfied.

- Thus, if the $T$ axis points upward, timelike curves remain 'steeper' than 45°, spacelike curves are 'flatter' than that, and causal relationships can be clearly read off the diagram.

*Interpretation*

The interpretation of the four regions (I - IV) in the analytically continued Schwarzschild spacetime (which is the region between the singularities in the Kruskal diagram 2.1) is as follows:

- The interior region bounded by the future horizon is what is called the interior of the Black Hole since no timelike or null curve starting from a point inside can lead out of the region. Therefore, neither matter nor information can leave this region (II).
- Similarly, region IV bounded by the past horizon is often referred to as a White Hole as every timelike or null curve starting from inside must leave the region in finite time. On the other hand, nothing can fall in from outside.
- There are indeed two causally disconnected exterior regions (I, III) which join together at the event horizon. Since the Kruskal diagram preserves the direction of null rays and thus the causal structure of spacetime, it is obvious from it that no timelike or null curve connects regions I and III.

  Sometimes, the exterior regions are charted in an embedding diagram (fig. 2.3, explanation is given there). The interior region is no longer part of the geometry covered by this diagram.
- In a Kruskal diagram, the exchange of the rôles of space and time in a Black Hole geometry mentioned earlier can be seen clearly.

## 2.2 Penrose diagrams

### 2.2.1 A simple example: Minkowski spacetime

*The simplest transformation*

In order to visualize an infinitely extended spacetime we perform a non-linear coordinate transformation which maps infinity to finite values of the new coordinates.

In spherically symmetric spacetimes we are concerned with here, the first step is to use spherical coordinates $t, r, \theta, \phi$ so there's only one spatial coordinate capable of assuming infinite values, namely $r$.

One possible function relating a finite to an infinite interval is the tangent function (fig. 2.4); the most naïve thing to do would be the transformation

$$t = \tan t' \ , \qquad r = \tan r' \ . \tag{2.11}$$

This already yields a compact portrait of spacetime (fig. 2.5) but on closer inspection, we find two 'flaws' we shall wish to overcome:

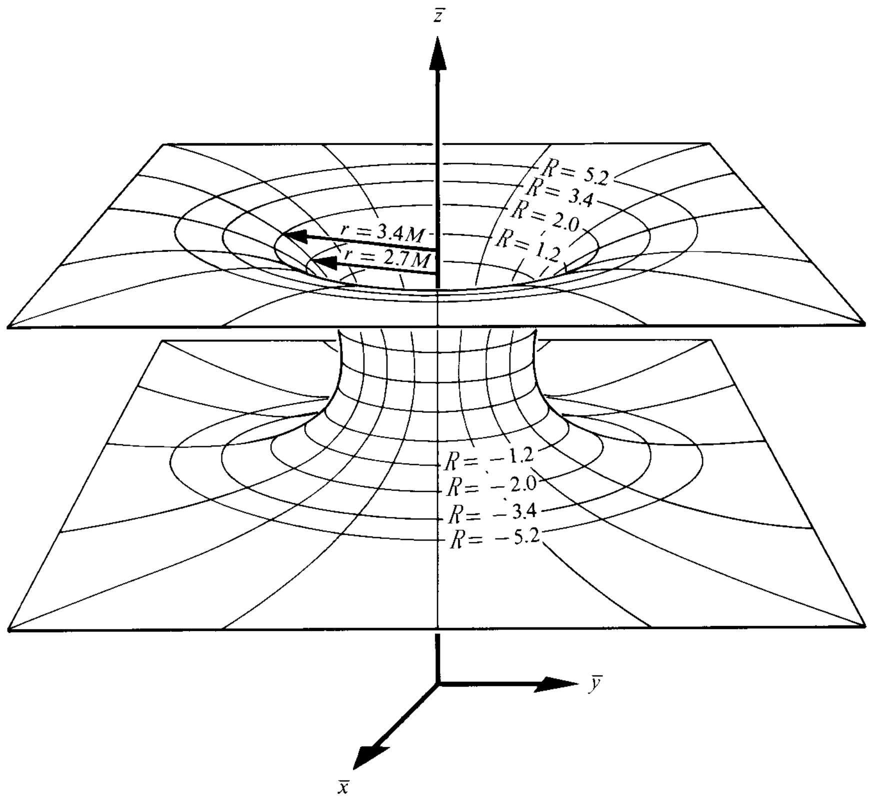


Figure 2.3: Embedding diagram of the exterior region of a Black/White Hole pair, joined together at their event horizons (taken from [1], fig. 31.5). Here, $t = T = 0$ and $\theta = \pi/2$. The interior geometry of the pictured surface is that of the $z = 0$ plane in the real world. The surface is given by revolving the hyperbola $\bar{z} = 2M + (\bar{x}^2 + \bar{y}^2)/8M$ about the $\bar{z}$ axis.

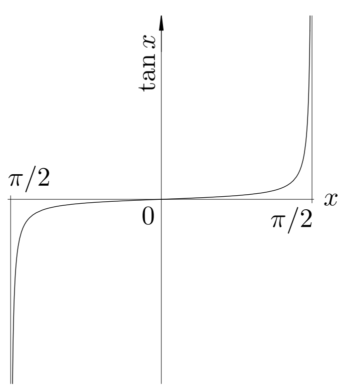


Figure 2.4: The tangent function relating the compact interval $[-\pi/2, \pi/2]$ to the infinite interval $(-\infty, \infty)$

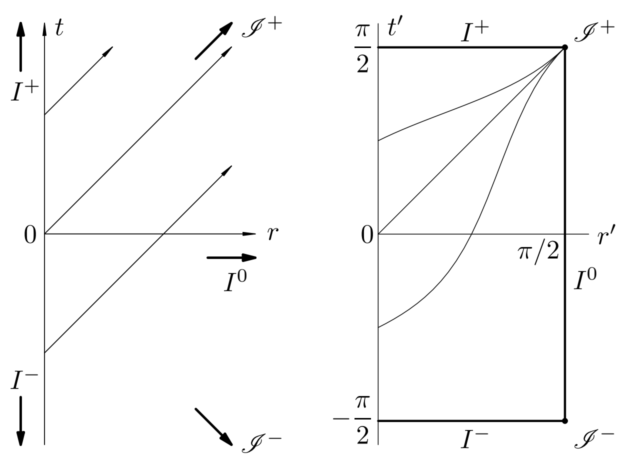


Figure 2.5: The compact portrait of $(t, r)$-space under the transformation (2.11). $I^0$, $I^{\pm}$, and $\mathscr{I}^{\pm}$ denote spatial, past and future, and past and future null infinity, resp. The same three null lines are drawn in both pictures.

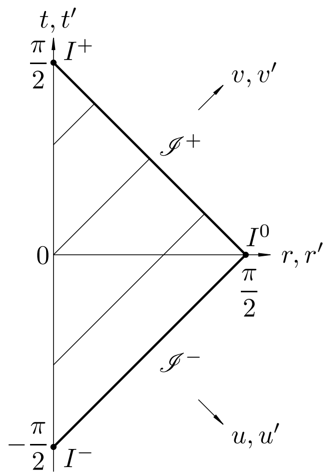


Figure 2.6: The compact portrait of $(t, r)$-space under the transformation (2.12)

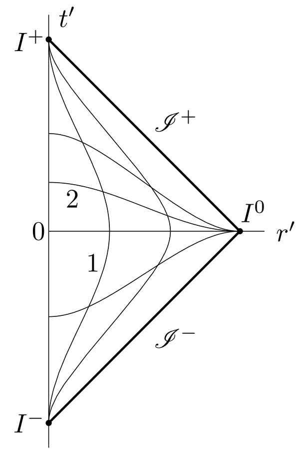


Figure 2.7: Lines of constant $r$ (1) and constant $t$ (2)

1. Null rays are not in general straight lines inclined by 45°. Only those passing through the origin are.

2. The metric is no longer conformally flat:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 = \frac{1}{\cos^4 t'}\,\mathrm{d}t'^2 - \frac{1}{\cos^4 r'}\,\mathrm{d}r'^2 - \tan^2 r'\,\mathrm{d}\Omega^2 \ .$$

*Null coordinates*

In our naïve model we observe that lines $t = \text{const}$ and $r = \text{const}$ remain straight lines in the horizontal and vertical directions, resp. Thus, the idea is to shrink the $(t, r)$-subspace not in the $r$- and $t$-directions but in the directions in which we want to keep lines straight. Let's introduce null coordinates

$$u = t - r \ , \qquad v = t + r$$

and perform a transformation

$$u = \tan u' \ , \qquad v = \tan v' \ . \tag{2.12}$$

Now the images of the several regions of infinity (time-like, space-like, null infinity) look different as compared to our naïve attempt (see fig. 2.6) but null rays $u' = 0$ and $v' = 0$ are straight lines inclined by 45° like in a plain spacetime diagram.

The price we pay for this is illustrated in fig. 2.7: Lines with $r = \text{const}$ or $t = \text{const}$ are not horizontal or vertical straight lines anymore, except for the coordinate axes themselves as they coincide with the old ones. The coordinates $r'$ and $t'$ used in the figures are defined by

$$t' = \frac{1}{2}(u' + v') \ , \qquad r' = \frac{1}{2}(v' - u') \ .$$

A look at the metric transformed according to (2.12) shows us another property of this transformation. It reads

$$\mathrm{d}s^2 = \mathrm{d}u\,\mathrm{d}v - \frac{(v-u)^2}{4}\,\mathrm{d}\Omega^2 = \frac{\mathrm{d}u'\,\mathrm{d}v'}{\cos^2 u'\,\cos^2 v'} - \frac{(\tan v' - \tan u')^2}{4}\,\mathrm{d}\Omega^2$$

which is not conformally flat, either, but its $(t, r)$-part is.

### 2.2.2 Black Holes

Finally we can apply the technique of compactifying an infinite manifold by means of conformal transformations to the Schwarzschild geometry, thereby obtaining Penrose diagrams of Black Holes.

Again we want null rays to remain at 45° with respect to the new time and space coordinates $T'$ and $R'$. This is ensured by the transformations

$$\frac{U}{4M} = \tan U' \ , \qquad \frac{V}{4M} = \tan V' \tag{2.13}$$

$$\text{and} \qquad T' = \frac{1}{2}(U' + V') \ , \qquad R' = \frac{1}{2}(V' - U') \ .$$

In terms of the new null coordinates, the line element reads

$$\mathrm{d}s^2 = \frac{2M}{r}\, e^{-r/2M}\, \frac{1}{\cos^2 U' \, \cos^2 V'}\, \mathrm{d}U' \, \mathrm{d}V' - r^2 \, \mathrm{d}\Omega^2$$

which has a conformally flat $(U', V')$ part again.

Transforming a whole $(U, V)$ plane according to (2.13) one would expect a diamond-shaped portrait (cf. fig. 2.6). However, only the part of the plane between the two $r = 0$ hyperbolae belongs to our spacetime manifold, so we ask the question: What becomes of those hyperbolae?

If $r = 0$, i.e. $r^* = 0$,

$$\frac{U}{4M}\frac{V}{4M} = \pm 1 \qquad \text{or} \qquad \tan U' = \cot V'$$

(see eqn. (2.8)). With both $U', V' \in [-\pi/2, \pi/2]$ this condition is satisfied if

$$U' + V' = \pm\frac{\pi}{2} \ .$$

Thus, the curves marking the singularity at $r = 0$ are mapped to straight lines parallel to the $R'$ axis (fig. 2.8).

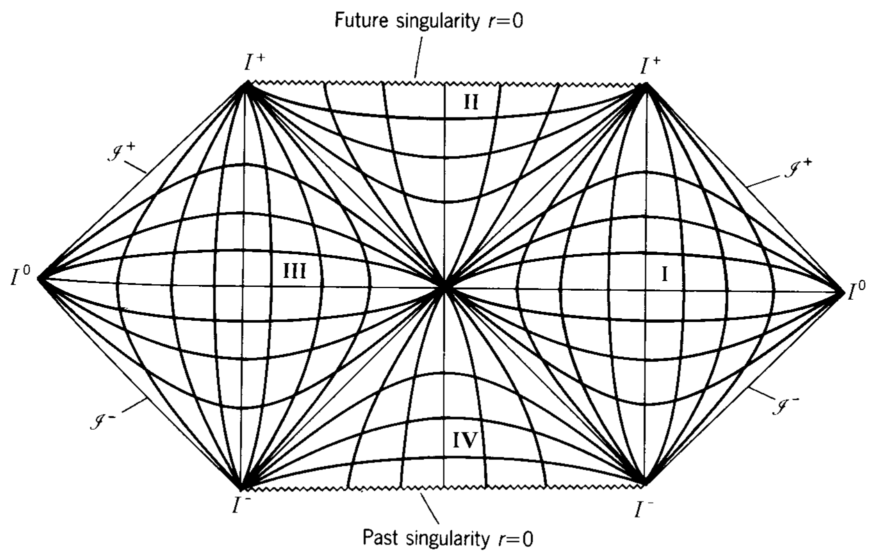


Figure 2.8: Penrose diagram for an ever-existing Black Hole (taken from [9]). Lines of constant $t$ are basically 'sideways' directed in regions I and III, lines of constant $r$ are basically 'upward' directed there, and vice versa in II and IV.

Chapter 3

# Particle Creation by Non-Minkowskian Spacetimes

## 3.1 Quantum fields and the particle concept

In this chapter, we are concerned with a phenomenon that doesn't arise from classical theories like General Relativity alone: the creation of particles defined as excitations of field modes.

It occurs in quantum field theory if one considers, e.g., external fields (including the curvature of spacetime by gravitation), boundary conditions, or non-inertial motion. Examples include cosmological particle creation, the Hawking and the Casimir effect, and – for accelerated motion – the Unruh effect.

We consider a scalar quantum field $\hat{\Phi}$.

### 3.1.1 Quantum fields and the wave equation

*The field equation and conformal transformations*

Scalar fields obey the Klein-Gordon equation

$$\left(\Box - m^2 + \xi R\right)\hat{\Phi} = \left(\frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\right) - m^2 + \xi R\right)\hat{\Phi} = 0\ . \tag{3.1}$$

Here, $m$ denotes the mass assigned to the field, $\Box$ is the curved spacetime d'Alembertian with metric $g_{\mu\nu}$, and $\xi$ determines the coupling of the matter field to the curvature of spacetime beyond that implied by the geometry dependence of the covariant derivatives involved.

There are two special cases of coupling: If $\xi = 0$, the field is said to be minimally coupled. For

$$\xi = \xi_c(n) = \frac{1}{4}\frac{n-2}{n-1}$$

where $n$ denotes the number of spacetime dimensions, we speak of conformal coupling. In two dimensions, minimal and conformal coupling are the same.

We speak of conformal coupling because for this kind of coupling, the field equation is essentially invariant under conformal transformations: A conformal transformation of the metric,

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = C(x)g_{\mu\nu}(x)\ , \qquad g^{\mu\nu}(x) \to \bar{g}^{\mu\nu}(x) = \frac{1}{C(x)}g^{\mu\nu}(x)\ ,$$

implies for the conformally coupled wave operator the relation

$$\Box + \xi_c(n)R(x) \to \overline{\Box + \xi_c(n)R(x)} = \big[C(x)\big]^{-(n+2)/4}\big(\Box + \xi_c(n)R(x)\big)\big[C(x)\big]^{(n-2)/4} \ . \quad (3.2)$$

On the other hand, the transformation of fields reads

$$\hat{\Phi} \to \bar{\hat{\Phi}} = \big[C(x)\big]^{-(n-2)/4}\hat{\Phi} \ .$$

Given the Klein-Gordon equation (3.1), these quantities satisfy

$$\begin{aligned}\big[C(x)\big]^{-(n+2)/4}\big(\Box + \xi_c(n)R - m^2\big)\hat{\Phi} &= 0\\ \big(\bar{\Box} + \xi_c(n)\bar{R}\big)\bar{\hat{\Phi}} - \big[C(x)\big]^{-(n+2)/4}m^2\big[C(x)\big]^{(n-2)/4}\bar{\hat{\Phi}} &\\ = \left(\bar{\Box} + \xi_c(n)\bar{R} - \frac{m^2}{C(x)}\right)\bar{\hat{\Phi}} &= 0\end{aligned}$$

where $\bar{R}$ is the Ricci scalar of the transformed metric $\bar{g}_{\mu\nu}$.

Thus, 'essentially invariant' means invariant up to a rescaling of the mass parameter. Obviously, a massless wave equation is conformally invariant.

In particular, this allows us to reduce the wave equation for a conformally flat metric to the one for Minkowski space.

*Decomposition of fields*

Let's decompose the field in normal modes $f_{\mathfrak{k}}$,

$$\hat{\Phi} = \int \mathrm{d}^3\mathfrak{k}\, \left(\hat{a}_{\mathfrak{k}} f_{\mathfrak{k}} + \hat{a}^\dagger_{\mathfrak{k}} f^*_{\mathfrak{k}}\right) \ . \quad (3.3)$$

The vector index $\mathfrak{k}$ counts the normal modes. It has the same dimension as space, $(n-1)$, and is, physically, the spatial wave vector of the modes.

The modes $f_{\mathfrak{k}}$ are normalized with respect to the Klein-Gordon scalar product defined by [4]

$$\langle\psi|\zeta\rangle = -i\int \sqrt{-g_\Sigma}\, n^\mu\, \psi\overleftrightarrow{\partial_\mu}\zeta^*\, \mathrm{d}\Sigma \quad (3.4)$$

where $\Sigma$ is supposed to be a spacelike hyper 3-surface, $n^\mu$ a future-directed unit vector field orthogonal to $\Sigma$, and $g_\Sigma$ is the determinant of the metric within $\Sigma$.

The choice of the scalar product (3.4) ensures that the inner product of two solutions to the Klein-Gordon equation (3.1) is conserved as the parameter time advances.

Orthonormalization can now be expressed in terms of the scalar product just defined:

$$\langle f_{\mathfrak{k}}|f_{\mathfrak{k}'}\rangle = \delta(\mathfrak{k} - \mathfrak{k}') \qquad \text{and} \qquad \langle f^*_{\mathfrak{k}}|f_{\mathfrak{k}'}\rangle = 0 \ . \quad (3.5)$$

### 3.1.2 Symmetries and the particle concept

In static regions where the space-time admits a Killing vector field which is timelike and hypersurface-orthogonal everywhere, the normal modes can be written in the form

$$f_{\mathfrak{k}}(x) = F_{\mathfrak{k}}(\mathfrak{x}) \cdot e^{i\omega(\mathfrak{k})x^0} \ .$$

Here, we have chosen coordinates such that the $x^0$ axis points in the direction of the Killing vector field and $\mathfrak{x}$ lies inside a hypersurface orthogonal to it, $x^0 = \text{const}$. Of course, the $x^0$ parameterization must be chosen appropriately.

By (3.3), this implies a unique decomposition of the field into creation and annihilation operators of excitations of each mode $\mathfrak{k}$. If the Killing vector field is a global one, then this decomposition is the same everywhere, and a vacuum $|0\rangle$ can be globally defined:

$$\hat{a}_{\mathfrak{k}}|0\rangle = 0 \quad \text{for all } \mathfrak{k}.$$

It is also possible to find a unique notion of vacuum and thus of particles in regions which permit a Killing field which is timelike everywhere, but not hypersurface-orthogonal [10]. Such regions are called stationary.

If no global Killing vector field exists, a vacuum must be defined by choosing local coordinate systems. It is not possible to choose them in such a way that the thus defined vacua agree for all $x$. Since the quantum state is a global concept, it is then impossible to define a global vacuum. If one observer locally detects vacuum, there is always another one who will see particles.

The same thing happens when an observer in a spacetime which at least locally admits a Killing vector field, doesn't follow it. An example is accelerated motion in flat space which allows for several global Killing vector fields which are, however, not Lorentz invariant. This phenomenon is known as the Unruh effect.

### 3.1.3 Asymptotic regions and Bogoliubov transformations

A special case of a spacetime that doesn't admit global Killing vector fields is a spacetime with asymptotic regions that do allow for Killing fields restricted to each of the regions. An example which is important for us is the Schwarzschild spacetime where the asymptotic regions are $\mathscr{I}^+$ and $\mathscr{I}^-$ (see fig. 2.8).

These Killing fields will, in general, not lead to the same vacuum state. Thus an observer who goes from one asymptotic region to another will register a changing occupation of the quantum state with particles. In particular, he will see particles being created out of an initial vacuum.

Mathematically, this means a mixing of annihilation and creation operators defined in each of the asymptotic regions. Annihilation operators of one region (1) will contain creation operators from the other (2) [4]:

$$\hat{a}_{2,\mathfrak{k}'} = \int \mathrm{d}^3k \big(\alpha_{\mathfrak{k}\mathfrak{k}'}\hat{a}_{1,\mathfrak{k}} + \beta^*_{\mathfrak{k}\mathfrak{k}'}\hat{a}^\dagger_{1,\mathfrak{k}}\big) \quad \text{and} \tag{3.6a}$$

$$\hat{a}_{1,\mathfrak{k}} = \int \mathrm{d}^3k' \big(\alpha^*_{\mathfrak{k}\mathfrak{k}'}\hat{a}_{2,\mathfrak{k}'} - \beta^*_{\mathfrak{k}\mathfrak{k}'}\hat{a}^\dagger_{2,\mathfrak{k}'}\big) \ . \tag{3.6b}$$

The same coefficients $\alpha_{\mathfrak{k}\mathfrak{k}'}$ and $\beta_{\mathfrak{k}\mathfrak{k}'}$ connect the sets of modes:

$$f_{2,\mathfrak{k}'} = \int \mathrm{d}^3k \big(\alpha^*_{\mathfrak{k}\mathfrak{k}'}f_{1,\mathfrak{k}} - \beta_{\mathfrak{k}\mathfrak{k}'}f^*_{1,\mathfrak{k}}\big) \quad \text{and} \tag{3.7a}$$

$$f_{1,\mathfrak{k}} = \int \mathrm{d}^3k' \big(\alpha_{\mathfrak{k}\mathfrak{k}'}f_{2,\mathfrak{k}'} + \beta_{\mathfrak{k}\mathfrak{k}'}f^*_{2,\mathfrak{k}'}\big) \ . \tag{3.7b}$$

This relation between different sets of modes as well as creation and annihilation operators is called a Bogoliubov transformation.

As a consequence of orthonormalization, the coefficients satisfy the following conditions:

$$\int \mathrm{d}^3\mathfrak{l}\big(\alpha_{\mathfrak{k}\mathfrak{l}}\alpha^*_{\mathfrak{k}'\mathfrak{l}} - \beta_{\mathfrak{k}\mathfrak{l}}\beta^*_{\mathfrak{k}'\mathfrak{l}}\big) = \delta(\mathfrak{k}-\mathfrak{k}')$$
$$\int \mathrm{d}^3\mathfrak{l}\big(\alpha_{\mathfrak{k}\mathfrak{l}}\beta_{\mathfrak{k}'\mathfrak{l}} - \beta_{\mathfrak{k}\mathfrak{l}}\alpha_{\mathfrak{k}'\mathfrak{l}}\big) = 0 \ .$$

A vacuum state in region (1) may contains particles defined in region (2):

$$\hat{a}_{2,\mathfrak{k}}|0_1\rangle = \int \mathrm{d}^3k'\beta^*_{\mathfrak{k}\mathfrak{k}'}\hat{a}^\dagger_{1,\mathfrak{k}'}|0_1\rangle = \int \mathrm{d}^3k'\beta^*_{\mathfrak{k}\mathfrak{k}'}|1_{1,\mathfrak{k}'}\rangle \ .$$

This yields the particle number in the mode $\mathfrak{k}$:

$$N_{\mathfrak{k}} = \int \mathrm{d}^3k' \, |\beta_{\mathfrak{k}\mathfrak{k}'}|^2 \ . \tag{3.8}$$

## 3.2 An example of particle production: the sudden expansion of the 'universe'

### 3.2.1 The metric

Following the lines of [4] we consider a metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\mathfrak{r}^2$$

which reads

$$\mathrm{d}s^2 = C(\eta)\big(-\mathrm{d}\eta^2 + \mathrm{d}\mathfrak{r}^2\big)$$

where $\eta$ is the conformal time coordinate defined by

$$t = \int^{\eta} a(\eta')\,\mathrm{d}\eta' \qquad \text{and} \qquad a^2(t) = C\big(\eta(t)\big) \ .$$

In [4], a continuous function $C(\eta)$ is considered whereas we want to calculate a simpler example which can be compared to the limit of the continuous one: a step function

$$C(\eta) = C_-\Theta(-\eta) + C_+\Theta(\eta) = C_- + (C_+ - C_-)\Theta(\eta)$$

describing a sudden expansion of space. Then

$$C_{,\eta}(\eta) = (C_+ - C_-)\delta(\eta) \ .$$

With $\eta_{\mu\nu}$ being the Minkowskian metric of signature $(-+++)$, our metric then reads

$$g^{\mu\nu}(\eta) = \frac{1}{C(\eta)}\eta^{\mu\nu} \qquad \text{or} \qquad g_{\mu\nu}(\eta) = C(\eta)\eta_{\mu\nu}$$

implying

$$\sqrt{-g} = C^2 \ , \qquad \sqrt{g^{(3)}} = C^{3/2} \ ,$$

and the curvature (Ricci) scalar (as obtained by using MAPLE)

$$R = \frac{3}{2}\frac{2C(\eta)_{,\eta\eta}C(\eta) - \big(C(\eta)_{,\eta}\big)^2}{C(\eta)^3} \ .$$

Not that this expression involves the square of a $\delta$ distribution. However, we will not need to do explicit calculations with $R$ at $\eta = 0$ as we will solve the wave equation only where $R$ is well-defined and match the solutions afterwards.

### 3.2.2 The Klein-Gordon equation

In our case, the Klein-Gordon-equation (3.1) reads

$$\left(\frac{1}{C^2}\partial_\mu(C\partial^\mu) - m^2 + \xi R\right)\hat{\Phi} = 0 \; . \tag{3.9}$$

For conformal coupling, (3.2) allows us to rewrite (3.9) as

$$\left(C^{-\frac{3}{2}}\Box_0 C^{\frac{1}{2}} - m^2\right)\hat{\Phi} = 0 \tag{3.10}$$

where

$$\Box_0 = -\partial_\eta^2 + \triangle$$

is the d'Alembertian in a Minkowskian metric.

We want to calculate mode functions $f_{\mathfrak{k}}$ and write them in the form [4]

$$f_{\mathfrak{k}}(\eta, \mathfrak{r}) = \frac{1}{\sqrt{(2\pi)^3}}\,\chi_{\mathfrak{k}} \cdot e^{i\mathfrak{k}\mathfrak{r}} \; . \tag{3.11}$$

Plugging $f_{\mathfrak{k}}$ into (3.10), we obtain

$$\left(C^{-\frac{3}{2}}\partial_\eta^2 C^{\frac{1}{2}} + C^{-1}\mathfrak{k}^2 - m^2\right)\chi_{\mathfrak{k}} \cdot e^{i\mathfrak{k}\mathfrak{r}} = 0 \; . \tag{3.12}$$

In order for $f_{\mathfrak{k}}$ to be a solution, it must be finite, and so must $\chi_{\mathfrak{k}}$. Thus, we know all terms in (3.12) to be finite except $\partial_\eta^2 C^{1/2}\chi_{\mathfrak{k}}$; for the equation to hold, this one must be finite as well, implying differentiability to first order of $C^{1/2}\chi_{\mathfrak{k}}$.

### 3.2.3 Solutions

*Solutions in each time region*

In the two regions $\eta \gtrless 0$ where $C_\pm = \text{const}$ we can solve this equation very easily,

$$\begin{aligned} \chi_{\pm,\mathfrak{k}} &= \nu_{\pm,\mathfrak{k}}\left(\alpha_{\pm,\mathfrak{k}} e^{i\omega_{\pm,\mathfrak{k}}\eta} + \beta_{\pm,\mathfrak{k}} e^{-i\omega_{\pm,\mathfrak{k}}\eta}\right) \\ \text{where} \qquad \omega_{\pm,\mathfrak{k}} &= \sqrt{\mathfrak{k}^2 - C_\pm m^2} \; . \end{aligned} \tag{3.13}$$

Here, we demand

$$|\beta_{\pm,\mathfrak{k}}|^2 - |\alpha_{\pm,\mathfrak{k}}|^2 = 1$$

and choose the normalization constant $\nu_{\pm,\mathfrak{k}}$ such that solutions $f_{\mathfrak{k}}$ are normalized with respect to the Klein-Gordon inner product (3.4).

We choose $\Sigma$ to be ordinary three-dimensional space at a fixed time $\eta$ such that $n^\mu = C^{-1/2}(-1, 0, 0, 0)$ and $-g_\Sigma = g^{(3)} = C^3$. Equation (3.4) now reads

$$\langle\psi|\zeta\rangle = -iC(\eta)\int \mathrm{d}^3\mathfrak{r}\,\psi\overleftrightarrow{\partial_0}\zeta^* \; . \tag{3.14}$$

Now we impose the orthonormality conditions (3.5) on solutions to the Klein-Gordon equation (3.9),

$$\langle f_{\pm,\mathfrak{k}}|f_{\pm,\mathfrak{k}'}\rangle = \delta(\mathfrak{k} - \mathfrak{k}') \qquad \text{and} \qquad \langle f^*_{\pm,\mathfrak{k}}|f_{\pm,\mathfrak{k}'}\rangle = 0 \; . \tag{3.15}$$

Applying (3.14) to $f_{\mathfrak{k}}$ and $f_{\mathfrak{k}'}$ as given by (3.11) and (3.13) and making use of

$$\frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathfrak{r}\, e^{i(\mathfrak{k}-\mathfrak{k}')\mathfrak{r}} = \delta(\mathfrak{k}-\mathfrak{k}')$$

we obtain

$$\langle f_{\mathfrak{k}}|f_{\mathfrak{k}'}\rangle = 2C_\pm \left|\nu_{\pm,\mathfrak{k}}\right|^2 \omega_{\pm,\mathfrak{k}}$$

which, by comparison to (3.15), yields

$$|\nu_{\pm,\mathfrak{k}}| = \frac{1}{\sqrt{2C_\pm\omega_{\pm,\mathfrak{k}}}} \qquad \text{or} \qquad \nu_{\pm,\mathfrak{k}} = \frac{1}{\sqrt{2C_\pm\omega_{\pm,\mathfrak{k}}}}e^{i\xi_{\mathfrak{k}}} \ , \qquad \xi_{\mathfrak{k}} \text{ real} \ . \tag{3.16}$$

Using (3.13) and (3.16) in (3.11), we can write down the solutions in the following way:

$$f_{\pm,\mathfrak{k}}(\eta,\mathfrak{r}) = \frac{1}{\sqrt{2(2\pi)^3 C_\pm\omega_{\pm,\mathfrak{k}}}}\left(\alpha_{\pm,\mathfrak{k}}e^{i\omega_{\pm,\mathfrak{k}}\eta} + \beta_{\pm,\mathfrak{k}}e^{-i\omega_{\pm,\mathfrak{k}}\eta}\right)\cdot e^{i\mathfrak{k}\mathfrak{r}+i\xi_{\mathfrak{k}}} \ . \tag{3.17}$$

*The Bogoliubov transformation*

We now want to perform the Bogoliubov transformation between the sets of solutions (3.17) in the two regions $\eta < 0$ and $\eta > 0$. With creation and annihilation operators defined according to (3.3),

$$\hat{\Phi} = \int \mathrm{d}^3\mathfrak{k}\left(\hat{a}_{\pm,\mathfrak{k}}f_{\pm,\mathfrak{k}} + \hat{a}^\dagger_{\pm,\mathfrak{k}}f^*_{\pm,\mathfrak{k}}\right) \ ,$$

the transformations (3.6) and (3.7) yield

$$\hat{a}_{+,\mathfrak{k}} = \int \mathrm{d}^3k'\left(\alpha^*_{\mathfrak{k}\mathfrak{k}'}\hat{a}_{-,\mathfrak{k}'} - \beta^*_{\mathfrak{k}\mathfrak{k}'}\hat{a}^\dagger_{-,\mathfrak{k}'}\right)$$

and

$$f_{+,\mathfrak{k}} = \int \mathrm{d}^3k'\left(\alpha_{\mathfrak{k}\mathfrak{k}'}f_{-,\mathfrak{k}'} + \beta_{\mathfrak{k}\mathfrak{k}'}f^*_{-,\mathfrak{k}'}\right) \ .$$

Since a single mode solution has a defined spatial frequency $\mathfrak{k}$, the coefficients are of the form

$$\alpha_{\mathfrak{k}\mathfrak{k}'} = \alpha_{\mathfrak{k}}\delta(\mathfrak{k}-\mathfrak{k}') \qquad \text{and} \qquad \beta_{\mathfrak{k}\mathfrak{k}'} = \beta_{\mathfrak{k}}\delta(\mathfrak{k}-\mathfrak{k}') \ .$$

If we assume that there is no wave propagating backwards in time for $\eta < 0$,

$$\chi_{-,\mathfrak{k}} = \nu_{-,\mathfrak{k}}e^{i\omega_{-,\mathfrak{k}}\eta} \ ,$$

we can identify the Bogoliubov coefficients with our $\alpha$ and $\beta$ from eqn. (3.13),

$$\alpha_{+,\mathfrak{k}} = \alpha_{\mathfrak{k}} \qquad \text{and} \qquad \beta_{+,\mathfrak{k}} = \beta_{\mathfrak{k}} \ .$$

In order to calculate them, we remember that $C^{1/2}\chi_{\mathfrak{k}}$ is continuous and differentiable everywhere, including $\eta = 0$:

$$\sqrt{C_-}\,\chi_{-,\mathfrak{k}}\Big|_{\eta=0} = \sqrt{C_+}\,\chi_{+,\mathfrak{k}}\Big|_{\eta=0} \quad \text{and} \quad \sqrt{C_-}\,\partial_\eta\chi_{-,\mathfrak{k}}\Big|_{\eta=0} = \sqrt{C_+}\partial_\eta\chi_{+,\mathfrak{k}}\Big|_{\eta=0} \ .$$

Plugging in $\chi_{\pm,\mathfrak{k}}$ at $\eta = 0$, we have

$$\sqrt{C_-}\frac{e^{i\xi_\mathfrak{k}}}{\sqrt{2C_-\omega_{-,\mathfrak{k}}}}e^{i\omega_{-,\mathfrak{k}}\eta}\bigg|_{\eta=0} = \sqrt{C_+}\frac{e^{i\xi_\mathfrak{k}}}{\sqrt{2C_+\omega_{+,\mathfrak{k}}}}\left(\alpha_{+,\mathfrak{k}}e^{i\omega_{+,\mathfrak{k}}\eta} + \beta_{+,\mathfrak{k}}e^{-i\omega_{+,\mathfrak{k}}\eta}\right)\bigg|_{\eta=0}$$

and

$$\frac{\sqrt{C_-}\,e^{i\xi_\mathfrak{k}}}{\sqrt{2C_-\omega_{-,\mathfrak{k}}}}(i\omega_{-,\mathfrak{k}})e^{i\omega_{-,\mathfrak{k}}\eta}\bigg|_{\eta=0} = \frac{\sqrt{C_+}\,e^{i\xi_\mathfrak{k}}}{\sqrt{2C_+\omega_{+,\mathfrak{k}}}}(i\omega_{+,\mathfrak{k}})\left(\alpha_{+,\mathfrak{k}}e^{i\omega_{+,\mathfrak{k}}\eta} - \beta_{+,\mathfrak{k}}e^{-i\omega_{+,\mathfrak{k}}\eta}\right)\bigg|_{\eta=0} .$$

This simplifies to

$$\begin{aligned}\frac{1}{\sqrt{\omega_{-,\mathfrak{k}}}} &= \frac{1}{\sqrt{\omega_{+,\mathfrak{k}}}}\left(\alpha_{+,\mathfrak{k}} + \beta_{+,\mathfrak{k}}\right) \qquad \text{and}\\ \sqrt{\omega_{-,\mathfrak{k}}} &= \sqrt{\omega_{+,\mathfrak{k}}}\left(\alpha_{+,\mathfrak{k}} - \beta_{+,\mathfrak{k}}\right) .\end{aligned}$$

The solutions to these equations (and thus, the Bogoliubov coefficients) are

$$\alpha_\mathfrak{k} = \frac{1}{2}\sqrt{\frac{\omega_{+,\mathfrak{k}}}{\omega_{-,\mathfrak{k}}}}\,\frac{\omega_+ + \omega_-}{\omega_+} \qquad \text{and} \qquad \beta_\mathfrak{k} = \frac{1}{2}\sqrt{\frac{\omega_{+,\mathfrak{k}}}{\omega_{-,\mathfrak{k}}}}\,\frac{\omega_+ - \omega_-}{\omega_+} . \tag{3.18}$$

### 3.2.4 Comparison to the results from the literature

In the referred book, a smooth expansion is considered instead of our metric jumping at $\eta = 0$. For a conformal scale factor

$$C(\eta) = C_- + (C_+ - C_-)\tanh(\rho\eta) ,$$

the Bogoliubov coefficients for a transformation between the regions $\eta \to -\infty$ and $\eta \to \infty$ are computed to be

$$\begin{aligned}\alpha_\mathfrak{k} &= \sqrt{\frac{\omega_{+,\mathfrak{k}}}{\omega_{-,\mathfrak{k}}}}\,\frac{\Gamma\left(1 - i\frac{\omega_{-,\mathfrak{k}}}{\rho}\right)\Gamma\left(-i\frac{\omega_{+,\mathfrak{k}}}{\rho}\right)}{\Gamma\left(-i\frac{\omega_{+,\mathfrak{k}}+\omega_{-,\mathfrak{k}}}{2\rho}\right)\Gamma\left(1 - i\frac{\omega_{+,\mathfrak{k}}+\omega_{-,\mathfrak{k}}}{2\rho}\right)} \qquad \text{and}\\ \beta_\mathfrak{k} &= \sqrt{\frac{\omega_{+,\mathfrak{k}}}{\omega_{-,\mathfrak{k}}}}\,\frac{\Gamma\left(1 - i\frac{\omega_{-,\mathfrak{k}}}{\rho}\right)\Gamma\left(i\frac{\omega_{+,\mathfrak{k}}}{\rho}\right)}{\Gamma\left(i\frac{\omega_{+,\mathfrak{k}}-\omega_{-,\mathfrak{k}}}{2\rho}\right)\Gamma\left(1 + i\frac{\omega_{+,\mathfrak{k}}-\omega_{-,\mathfrak{k}}}{2\rho}\right)}\end{aligned}$$

where

$$\omega_{\pm,\mathfrak{k}} = \sqrt{\mathfrak{k}^2 - C_\pm m^2} .$$

Actually, only two dimensions are considered so the authors have $k$ instead of $\mathfrak{k}$ but the result ought to be the same.

We are interested in the limit $\rho \to \infty$; using MAPLE, we obtain

$$\lim_{\rho\to\infty}\frac{\Gamma(ia/\rho)}{\Gamma(ib/\rho)} = \frac{b}{a} .$$

With $\Gamma(1) = 1$, this leads to our results (3.18).

Chapter 4

# The Hawking Effect

In the last chapter, we gave an example for particle production by a spacetime different from Minkowski spacetime. There, the geometry changed with the time coordinate, in a very simple way. Basically the same effect is possible in a spacetime with a metric that changes along arbitrary worldlines. Vacuum has then to be defined in asymptotic regions that are characterized other than by their time parameter.

One such spacetime is that of a Black Hole; a Black Hole metric depends on the radial coordinate $r$, and the asymptotic regions are $\mathscr{I}^+$ and $\mathscr{I}^-$, see fig. 2.8.

If, in particular, the quantum state is such that there is vacuum at $\mathscr{I}^-$, it will correspond to a thermal ensemble of particles at $\mathscr{I}^+$ with the Hawking temperature $T_{\mathrm{H}}$. This effect is called Hawking radiation.

There is a variety of ways to explain the phenomenon of radiating Black Holes. We will consider three of them:

- The first one allows us to estimate (but not strictly derive) $T_{\mathrm{H}}$ in the picture of pairs of virtual particles torn apart by the Black Hole's gravitational field [11].
- Then we will consider the wave equation and make an ansatz for its solution by which we can reduce it to a Schrödinger-type wave equation with a complicated scattering potential [12]. If we could solve it explicitly, we could proceed as in the previous chapter.
- Last, there will be an outline of Hawking's own explanation of the Black Hole radiation [3].

## 4.1 The virtual particle picture

### 4.1.1 Separation of virtual particle pairs

It is a well-known quantum mechanical result that no particle can be at rest at a precisely given position, or alternatively have exactly zero energy at a precisely given moment of time. The reason for this is the uncertainty principle.

According to the same principle, quantum field theory doesn't allow for a perfect vacuum in the sense that a field mode has zero energy at any exactly given moment of time. Field modes can be in their ground state; if all the modes of a field are, this is referred to as 'vacuum'. But the uncertainty principle, one form of which is

$$\Delta E \cdot \Delta t \geq 1 \ , \tag{4.1}$$

allows the mode to be in an excited state at $\Delta E$ above the ground state for the time $\Delta t$, but not longer.

One can imagine these fluctuations as producing particles of energies $\Delta E$ and $-\Delta E$, resp., which have to recombine after a time $\Delta t$ at the longest.

There is no admissible trajectory for the negative energy particle outside the Black Hole. Consider a radially moving particle and denote the three-momentum $\mathfrak{p} = (p^1, p^2, p^3)$. Then, the norm of the four-momentum is

$$p^\mu p_\mu = -\left(1 - \frac{2M}{r}\right)(-\Delta E) + \frac{1}{1 - 2M/r}(p^1)^2 \ . \tag{4.2}$$

If $r > 2M$, this quantity is positive whereas it is bound to be negative in order for the particle to have a timelike trajectory.

However, since the geometry in the vicinity of the event horizon of a Black Hole is well-behaved, the negative energy particle can cross the horizon during this interval. Inside the event horizon, the expression given by eqn. (4.2) is certainly negative. The particle can propagate freely towards the central singularity as space and time exchange their rôle with regard to propagation towards increasing proper time, see chapter 2.

As the negative energy particle is no longer denied free motion, the particles become real. The remaining positive energy particle is free to escape to infinity. By this process, the Black Hole effectively radiates off the energy $\Delta E$.

### 4.1.2 Estimating the Hawking temperature

We will use the uncertainty relation (4.1) to obtain the average energy, $\Delta E$, of the particle escaping to infinity. In order to do this, we need an estimation for the time the particle exists in the classically forbidden state.

Starting a distance $\varepsilon$ away from the horizon, it takes a proper time interval

$$\Delta\tau = -\int_{2M+\varepsilon}^{2M} \frac{\mathrm{d}r}{\sqrt{\frac{2M}{r} - \frac{2M}{2M+\varepsilon}}} \approx 2\sqrt{2M\varepsilon}$$

to reach the horizon [11]. With this value, we obtain from the uncertainty relation

$$\Delta E = E_{r+\varepsilon} = \frac{1}{2\sqrt{2M\varepsilon}}$$

for the energy of the particle at a distance $\varepsilon$ from the horizon. It is related to the energy of the same particle at infinity by [11]

$$E_\infty = E_{r+\varepsilon}\sqrt{\frac{\varepsilon}{2M}} \ .$$

This yields an energy at infinity, i.e., the energy measured by the observer who investigates the Hawking radiation, of

$$E_\infty = \frac{1}{8\pi M} \ .$$

If we define the temperature of the radiation by $T = E$ where $E$ is the average energy of the detected Hawking particles, and Boltzmann's constant, $k_{\mathrm{B}}$, has been put equal to 1, we end up with a temperature

$$T = \frac{1}{8\pi M}$$

which coincides with the Hawking temperature obtained in the strict derivation of the Hawking effect.

These considerations do, however, not explain why the Black Hole radiation has the spectrum of black body radiation.

## 4.2 The curvature of spacetime as a scattering potential

Let's now consider waves instead of particles. For the problem at hand, this is the better approach by far as the very particle concept is ill-defined in a generally curved spacetime.

### 4.2.1 The s-wave equation for four-dimensional Black Holes

We restrict ourselves to spherically symmetric waves, called s-waves. This allows us to suppress the angular degrees of freedom in the calculations and provides a bridge to the two-dimensional models considered in the second part of this work.

Furthermore, we will talk only about a massless scalar field. This simplifies calculations considerably and suffices as the purpose of this section is just to demonstrate qualitatively the relation between particle production by a Black Hole and the scattering of a wave as considered in quantum mechanics.

As we deal with the exterior of four-dimensional Schwarzschild Black Holes in this chapter, the curvature scalar $R$ vanishes identically, thus rendering the distinction between minimally and conformally coupled fields unnecessary.

In the setup just outlined, the Klein-Gordon equation reads

$$\Box\hat{\Phi} = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\right)\hat{\Phi} = 0\ . \tag{4.3}$$

In the s-wave approximation, derivatives with respect to the angular variables vanish; thus eqn. (4.3) becomes

$$\partial_t\big(\sqrt{-g}\,g^{tt}\partial_t\Phi_s\big) + \partial_{r^*}\big(\sqrt{-g}\,g^{r^*r^*}\partial_{r^*}\Phi_s\big) = 0 \tag{4.4}$$

where we have already chosen tortoise coordinates (2.5). In these coordinates, we have

$$\begin{aligned} \mathrm{d}s^2 &= \left(1-\frac{2M}{r}\right)\left(-\mathrm{d}t^2+\mathrm{d}r^{*2}\right) - r^2\,\mathrm{d}\Omega^2\ , \\ -g^{tt} &= g^{r^*r^*} = \frac{1}{1-2M/r}\ , \\ \text{and}\quad \sqrt{-g} &= \left(1-\frac{2M}{r}\right)r^2\sin\theta\ . \end{aligned}$$

Plugging this into (4.4), dividing by $\sin\theta$, and using (2.7), we get

$$-r^2\partial_t^2\Phi_s + \partial_{r^*}\left(r^2\partial_{r^*}\Phi_s\right) = 0\ ,$$

$$-r^2\partial_t^2\Phi_s + r^2\partial_{r^*}^2\Phi_s + 2r\left(1-\frac{2M}{r}\right)\partial_{r^*}\Phi_s = 0 \tag{4.5}$$

### 4.2.2 Reduction to a scattering problem

We consider spherically symmetric normal modes $f_i(t, r^*)$ and make a separation ansatz

$$f_i(t, r^*) = \frac{1}{r} T(t) R(r^*) \ .$$

Dividing (4.5) by $r$, we have

$$R \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} - rT \frac{\mathrm{d}^2}{\mathrm{d}r^{*2}} \frac{R}{r} - 2T \left(1 - \frac{2M}{r}\right) \frac{\mathrm{d}}{\mathrm{d}r^*} \frac{R}{r} = 0$$

and dividing further by $RT$ yields

$$\frac{1}{T} \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = \frac{r}{R} \frac{\mathrm{d}^2}{\mathrm{d}r^{*2}} \frac{R}{r} + \frac{2}{R} \left(1 - \frac{2M}{r}\right) \frac{\mathrm{d}}{\mathrm{d}r^*} \frac{R}{r} = \omega^2 = \text{const} \tag{4.6}$$

since each of the expressions is independent of one of the variables $t, r^*$. Therefore the time-dependent part of the solution is a harmonic oscillation,

$$T(t) = T_1 e^{-i\omega t} + T_2 e^{i\omega t} \ , \qquad T_1, T_2 = \text{const} \ .$$

Normal modes are counted by the continuous parameter $\omega$ which we assume to be positive.

Using

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r^*} \frac{R}{r} &= \frac{1}{r} \frac{\mathrm{d}R}{\mathrm{d}r^*} - \frac{1}{r^2} \left(1 - \frac{2M}{r}\right) R \qquad \text{and} \\ \frac{\mathrm{d}^2}{\mathrm{d}r^{*2}} \frac{R}{r} &= \frac{1}{r} \frac{\mathrm{d}^2 R}{\mathrm{d}r^{*2}} - \frac{2}{r^2} \left(1 - \frac{2M}{r}\right) \frac{\mathrm{d}R}{\mathrm{d}r^*} + \left(\frac{2}{r^3} - \frac{6M}{r^4}\right) \left(1 - \frac{2M}{r}\right) R \end{aligned}$$

the $r^*$-dependent part of (4.6) reduces to

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^{*2}} + \left[\omega^2 - \left(1 - \frac{2M}{r}\right) \frac{2M}{r^3}\right] R = 0 \ . \tag{4.7}$$

This reminds us of a potential scattering problem as known from quantum mechanics, only with a very complicated potential. The wave $R$ depends on $r^*$ while the potential depends on $r$ which in turn is a transcendental function of $r^*$. It is the so-called Lambert W function which is available only tabularized.

### 4.2.3 Discussion

Finally, let's look at the field modes we were just talking about in a Kruskal diagram (see fig. 2.1) or Penrose diagram (see fig. 2.8) of Schwarzschild spacetime.

We have a wave $R_{\text{in}}$ incident from $r^*, r \to \infty$ (the wave coming from $\mathscr{I}^-$) which is partly transmitted to the region $r^* \to -\infty, r \to 2M$ ($R_{BH}$, swallowed by the Black Hole), and partly reflected back off the potential wall to $r^*, r \to \infty$ ($R_{\text{out}}$, having the same frequency as $R \in$ as $V$ is the same). No wave is incident from $r^* \to -\infty$ since we assume the White Hole located there not to emit. See figure 4.1.

In the region $r^* \to \infty$, eqn. (4.7) reduces further to a harmonic oscillator equation with wave number $k = \omega$,

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^{*2}} = \omega^2 R \ ,$$

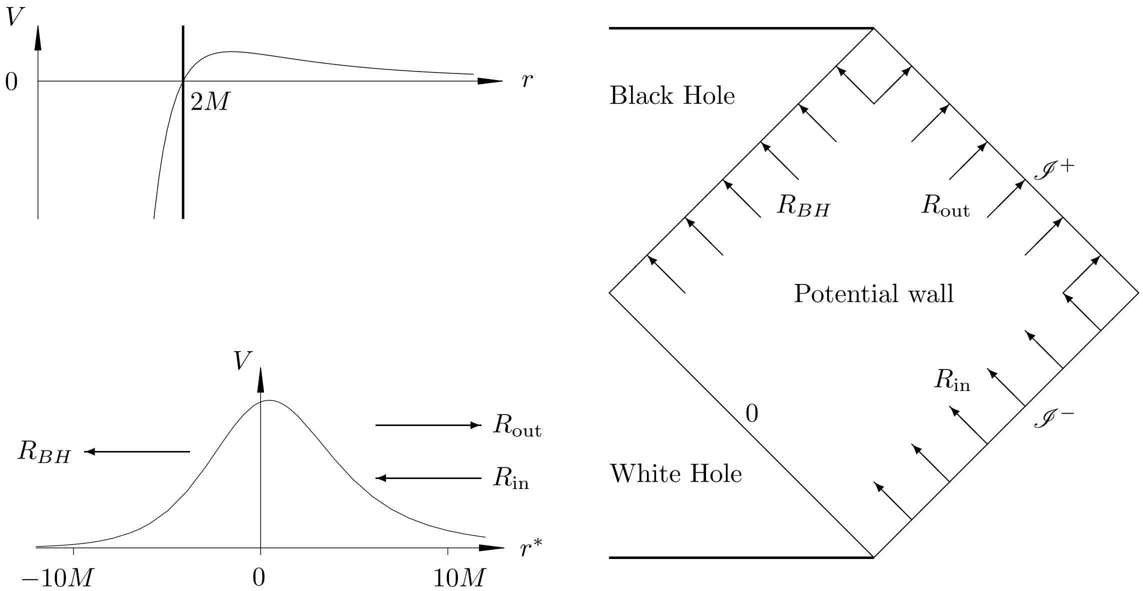


Figure 4.1: *Left:* The potential $V = \left(1 - \dfrac{2M}{r}\right)\dfrac{2M}{r^3}$ as a function of $r$ and $r^*$, resp. *Right:* The wave $R_{\text{in}}$ incident from $\mathscr{I}^-$ is partly swallowed by the Black Hole ($R_{BH}$) and partly reflected to $\mathscr{I}^+$ ($R_{\text{out}}$) while the White Hole doesn't emit.

$$R(r^*) = R_1 e^{-ikr^*} + R_2 e^{ikr^*} \ , \qquad R_1, R_2 = \text{const} \ .$$

Far from the singularity we thus have plane waves of asymptotically constant wavelength with respect to both $r$ and $r^*$.

On the other hand, near the horizon the wavelength is asymptotically constant with respect to $r^*$ as well but, as $r^* \to -\infty$ and $r \to 2M$, more and more wave fronts pile up against the horizon the closer one gets to it, thus infinitely shortening the wavelength, and the wave is blue-shifted as seen by an observer falling freely into the Black Hole (who crosses the horizon in finite proper time, just as the $r$ coordinate runs through a finite interval while doing so).

## 4.3 Hawking's derivation of Black Hole radiation

### 4.3.1 Gravitational collapse

In order to follow the derivation of Black Hole radiation as given by Stephen Hawking himself, we need to consider the gravitational collapse of a star into a Black Hole, as opposed to eternal Black Holes.

We will consider a star which is supposed to consist of spherically symmetrically distributed matter, without making any further assumptions on the nature or radial distribution of that matter. It will turn out to be of no importance for the Hawking process.

By Birkhoff's theorem (see, e.g., [1]) the gravitational field at some point within such a distribution of matter depends only on the total mass contained inside a sphere concentric with the distribution and possessing a surface that contains the point in question. All matter outside that sphere gives rise to zero total gravitational influence.

Applied to our case, this means that outside a collapsing star, the gravitational field

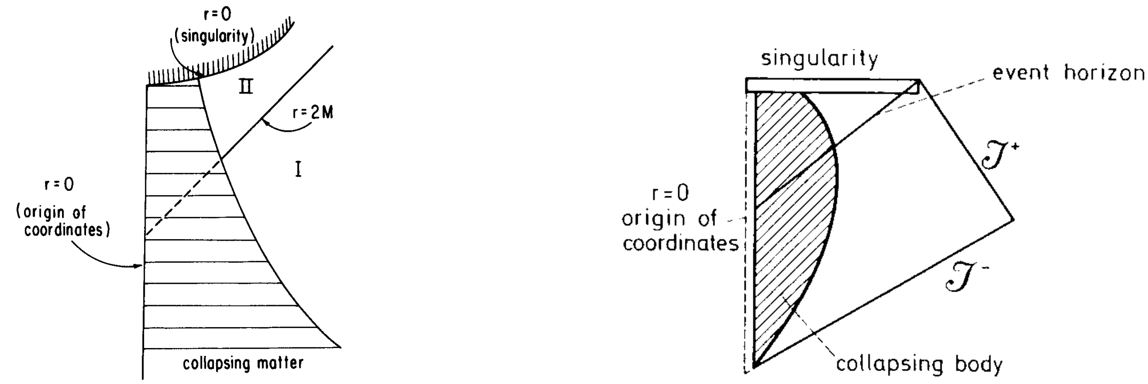


Figure 4.2: *Left:* Kruskal diagram and *right:* conformal diagram of the gravitational collapse of a star. Its interior is marked by hatching. Figures taken from [13] and [3], resp.

is always that of Schwarzschild spacetime. Points on the surface itself follow trajectories through the spacetime of a Black Hole of mass $M$ which is the total mass of the star.

In order to draw a spacetime diagram of gravitational collapse, one can take the diagram belonging to the $(t, r)$ part of an eternal Black Hole and ‘cut out’ everything that lies on the Black Hole’s side of a radial geodesic assigned to some particle that falls inwards from infinity, as we let – for simplicity – our star collapse from infinity. See figure 4.2, left.

To draw the corresponding Penrose diagram (see fig. 4.2, right), we apply a conformal transformation different from that introduced in chapter 2. It is the same only as far as the exterior region is concerned, but transforms that part of the line $r = 0$ which is not yet a singularity into a straight vertical line. However, the lines representing the hypersurfaces $\mathscr{I}^+$ and $\mathscr{I}^-$ remain unchanged by this procedure, as do the paths of light rays anywhere inside the diagram.

### 4.3.2 Black Hole radiation

The light rays coming from $\mathscr{I}^-$ or going to $\mathscr{I}^+$ mark lines of constant phase of a normal mode of the matter field propagating along $\mathscr{I}^-$ and $\mathscr{I}^+$, resp. Therefore, if we map a point $P^+$ on $\mathscr{I}^+$ to a point $P^-$ on $\mathscr{I}^-$ by following the light ray coming out on $\mathscr{I}^+$ at $P^+$ back to $\mathscr{I}^-$, the phase of a normal mode considered at both points will be the same. By taking into account small vicinities of the points $P^+$ and $P^-$, we can compare the frequency of the mode along $\mathscr{I}^+$ with that along $\mathscr{I}^-$. This is what we need in order to perform the Bogoliubov transformation between the modes on $\mathscr{I}^+$ and $\mathscr{I}^-$ and, via the Bogoliubov coefficients, calculate the radiation from the Black Hole assuming there is an incoming vacuum.

As our spacetime diagrams contain a radial coordinate, we have to reflect a light ray going through the origin from the line $r = 0$. Doing so in the Penrose diagram in fig. 4.2, we see that all the rays arriving at $\mathscr{I}^+$ come from that part of $\mathscr{I}^-$ with $v < v_0$. Light rays at later advanced times $v$ are swallowed by the Black Hole. As $\mathscr{I}^+$ is an infinite interval and $v_0$ is a finite coordinate value, wave crests equidistant on $\mathscr{I}^+$ pile up near $v_0$ when followed back to $\mathscr{I}^-$. This means very high frequencies which retrospectively justifies using the geometrical optics approximation made when considering light rays that pass through the interior of the collapsing star undisturbed.

As $\mathscr{I}^+$, $\mathscr{I}^-$, and all light rays are at angles of 45° in our figure, distances between two particular wave crests measured in the diagram are the same at $\mathscr{I}^+$ and at $\mathscr{I}^-$. So we need to know the relation between the coordinates used in the diagram and the actual $u$ and $v$ coordinates. This relation is given by the conformal transformation applied when drawing the diagram. For $\mathscr{I}^+$, it is given by eqn. (2.13).

Thus, we can compare outgoing normal modes of frequency $\omega$ defined on $\mathscr{I}^+$ – with respect to $u$ – to incoming ones with frequency $\omega'$ defined on $\mathscr{I}^-$ – with respect to $v$ – and propagated to $\mathscr{I}^+$. A part of each mode will be reflected off the static gravitational potential outside (static because of Birkhoff's theorem, see above) and arrive at $\mathscr{I}^+$ with the frequency $\omega'$. The part going through the interior of the mass distribution experiences a changing potential and thus will be effectively red-shifted. For this latter part, Hawking [3] obtains Bogoliubov coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ which satisfy the relation

$$|\alpha_{\omega\omega'}| = e^{\pi\omega/\kappa}|\beta_{\omega\omega'}|$$

where $\kappa = 1/4M$ is the surface gravity of the Black Hole of mass $M$. The approximation made here is to consider incoming modes to have the asymptotic form valid near $v_0$ even at earlier advanced times $v$.

Demanding, as in the example in the previous chapter,

$$|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2 = 1 \ ,$$

we obtain

$$|\beta_{\omega\omega'}|^2 = \frac{1}{e^{2\pi\omega/\kappa} - 1} \ . \tag{4.8}$$

By eqn. (3.8), the number of particles defined with respect to the outgoing mode of frequency $\omega$ is infinite which corresponds to a steady finite emission rate that remains forever after the gravitational collapse. However, expression (4.8) describing the ratio of particle emission in different outgoing modes corresponds to thermal radiation at the Hawking temperature

$$T_{\mathrm{H}} = \frac{1}{8\pi M} \ .$$

In [3], Hawking does not consider particles as excitation of field modes but as wave packets instead. Both approaches lead to the same thermal distribution at the Hawking temperature.

# Part II

# *Two-Dimensional Models & the Effective Action Approach*

## Chapter 5

# Two-Dimensional Effective Action Models

### 5.1 Why consider two-dimensional models?

Black Holes lose mass by the process of Hawking radiation which we reviewed in the previous chapter. The emission rate is the higher the higher the Hawking temperature of the Hole, i.e., the smaller its mass. This implies a permanent acceleration of the emission process which is probably destined to end in an outburst of radiation that destroys the Black Hole.

Simple as this might sound, it is not possible today to strictly calculate this process of a Black Hole's evaporation.

The full theory needed to describe the evolution of a radiating Black Hole is a theory of quantum gravity which is, however, not yet available. The next thing to work with is a semiclassical approximation: matter is supposed to obey quantum laws where the geometry of spacetime is treated as a classical one which, in turn, is related to the expectation value of the matter energy-momentum tensor by the semiclassical Einstein equations,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \langle T_{\mu\nu} \rangle \ . \tag{5.1}$$

Whether such an approach makes sense is subject to debate [4]. Due to the non-linearity of gravity, it will certainly fail for effects that occur on the scale of the Planck length or involve singularities. Thus it will certainly not be possible to correctly describe, among other things, the very final stage of Black Hole evaporation in a semiclassical model. On the other hand, one might expect meaningful results as long as one stays in the region exterior of a reasonably sized Black Hole. It is hoped that the semiclassical approximation in gravity works similarly to the quantum electromagnetic one which is, e.g., able to describe particles in exterior electromagnetic fields.

Yet in the semiclassical approximation as well as in full quantum gravity, the equations describing the evolution of the system must be solved self-consistently. In four dimensions, this poses a problem: One is only able to calculate, e.g., the Hawking radiation for a fixed background metric such as a Black Hole created by gravitational collapse. Little success has been made so far to include the effect of the back-reaction of Black Hole emission on the geometry.

In order to get an idea of what the back-reaction might look like, one considers two-dimensional models as they are easier to deal with. One reason for this is that every two-dimensional metric is conformally flat. In terms of null coordinates $u$ and $v$ it reads

$$ds^2 = -e^{2\sigma}\, du\, dv \ , \qquad \sqrt{-g} = \frac{1}{4}e^{2\sigma} \qquad \text{where } \sigma = \sigma(u, v) \ . \tag{5.2}$$

This is an important property to be used in two-dimensional calculations. Probably these models cannot yield more than qualitative suggestions as to the result of the four-dimensional case. However, even to achieve qualitative agreement with respect to the Hawking radiation on a fixed geometry is still an open problem.

The second part of this work is dedicated to two-dimensional model spacetimes, the problem of qualitative agreement between two- and four-dimensional Hawking radiation [6, 7], and an ansatz for a possible solution made by Y. Gusev and A. Zelnikov [8].

## 5.2 Action principle and conformal trace anomaly

### 5.2.1 The action principle

*The gravitational part*

If we build our theory on an action principle, we obtain the quantities of interest by variation of an action functional. This functional cannot be derived but rather must be assumed. The equations which describe the evolution of the system under consideration reflect the requirement that the action functional, evaluated on a given field configuration, be extremal.

The action $S$ describing a relativistic field theory consists of two parts: the gravitational action $S_{\rm g}$, and the matter action $S_{\rm m}$:

$$S = S_{\rm g} + S_{\rm m} \ .$$

It is possible in four dimensions to directly derive equation (5.1) from the action $S$. In order to do this, we have to decide on an explicit expression for $S_{\rm g}$. In four dimensions we choose

$$S_{\rm g}^{(4)} = \frac{1}{16\pi} \int {\rm d}^4x \, \sqrt{-g^{(4)}}\, R^{(4)} \ . \tag{5.3}$$

This action is known as the Einstein-Hilbert action [1]. Its variation with respect to the metric yields the left-hand side of eqn. (5.1).

*The matter part*

The right-hand side is not further specified in eqn. (5.1). Depending on the kind of matter under consideration, an action $S_{\rm m}$ is chosen and the expectation value of the energy-momentum tensor is derived from it by variation with respect to the metric.

For a minimally coupled, massless, non-self-interacting scalar matter field $\Phi$ in four-dimensional spacetime, the action $S_{\rm m}^{(4)}$ reads

$$S_{\rm m}^{(4)} = -\frac{1}{(4\pi)^2} \int {\rm d}^4x \, \sqrt{-g^{(4)}}\, (\nabla\Phi)^2 \ . \tag{5.4}$$

The classical energy-momentum tensor is now obtained in the following way,

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{\rm m}}{\delta g_{\mu\nu}} \ , \qquad T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\rm m}}{\delta g^{\mu\nu}} \tag{5.5}$$

which follows from the definition of the action,

$$\delta S_{\mathrm{m}} = -\int \mathrm{d}^2x \, \frac{\sqrt{-g(x)}}{2} \, T^{\mu\nu}(x) \delta g_{\mu\nu}(x) \ . \tag{5.6}$$

The change of sign in eqn. (5.5) is due to the way contravariant metric components are varied with respect to covariant ones, see eqn. (1.18) below.

In quantum field theory, a relation corresponding to eqn. (5.5) can be formally formulated for the vacuum expectation value of the energy-momentum tensor:

$$\langle T^{\mu\nu} \rangle = \frac{-2}{\sqrt{-g}} \frac{\delta S_{\mathrm{e}}}{\delta g_{\mu\nu}} \ , \qquad \langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{e}}}{\delta g^{\mu\nu}} \ , \tag{5.7}$$

$$\delta S_{\mathrm{e}} = -\int \mathrm{d}^2x \, \frac{\sqrt{-g(x)}}{2} \, \langle T^{\mu\nu}(x) \rangle \delta g_{\mu\nu}(x) \ . \tag{5.8}$$

Here, $S_{\mathrm{e}}$ is the effective action introduced below.

*The energy-momentum tensor in two and four dimensions*

The authors of [14] give a simple relation between the energy-momentum tensor in two dimensions and the one in four dimensions assuming spherical symmetry,

$$\langle T^{(4)}{}_{\mu\nu} \rangle = \frac{\langle T^{(2)}{}_{\mu\nu} \rangle}{4\pi r^2} \ .$$

However, there is also the possibility to determine the angular components of the four-dimensional energy-momentum tensor. In four dimensions, we vary the action with respect to the angular components of the metric. These do not occur in a two-dimensional model. The information on the angular metric components is contained in the dilaton field $\phi$ and the knowledge about spherical symmetry instead. With these, we obtain the tangential pressure [14]

$$\langle T^{(4)}{}^{\theta}_{\theta} \rangle = \langle T^{(4)}{}^{\varphi}_{\varphi} \rangle = \frac{1}{8\pi r^2 \sqrt{-g^{(2)}}} \frac{\delta S_{\mathrm{m}}}{\delta \phi} \ . \tag{5.9}$$

The $(\theta, \theta)$ and $(\varphi, \varphi)$ components are equal because of spherical symmetry.

### 5.2.2 Conformal invariance and its breaking

*The conformally flat two-dimensional case*

If we work in two dimensions and the metric $g_{\mu\nu}$ is related to the flat metric $\eta_{\mu\nu}$ by a conformal transformation,

$$g_{\mu\nu}(x) = e^{\sigma(x)} \eta_{\mu\nu}(x) \ , \qquad g^{\mu\nu}(x) = e^{-\sigma(x)} \eta^{\mu\nu}(x) \ , \tag{5.10}$$

eqn. (5.8) can be written

$$\begin{aligned} \delta S_{\mathrm{m}} &= -\int \mathrm{d}^2x \, \frac{\sqrt{-g}}{2} \langle T^{\mu\nu} \rangle (\delta\sigma) e^{\sigma} \eta_{\mu\nu} = -\int \mathrm{d}^2x \, \frac{\sqrt{-g}}{2} \langle T^{\mu\nu} \rangle (\delta\sigma) g_{\mu\nu} \\ &= -\int \mathrm{d}^2x \, \frac{\sqrt{-g}}{2} \langle T^{\mu}{}_{\mu} \rangle \delta\sigma \ . \end{aligned}$$

Since every two-dimensional metric can be represented in the form (5.10), the trace of the classical energy-momentum tensor in two dimensions is given by

$$T^\mu_\mu = \frac{-2}{\sqrt{-g}} \frac{\delta S_\mathrm{m}}{\delta \sigma} \ , \tag{5.11}$$

and, accordingly, its quantum counterpart is given formally by

$$\langle T^\mu_\mu \rangle = \frac{-2}{\sqrt{-g}} \frac{\delta S_\mathrm{m}}{\delta \sigma} \tag{5.12}$$

where the action $S_\mathrm{m}$ is, in contrast to the one in the previous equation, the action of the quantum matter field.

*The conformal anomaly*

Classically, a conformally invariant action yields, by eqn. (5.11), a zero trace of the energy-momentum tensor. If the theory in question does not contain any kind of length scale, the action may be invariant as conformal transformations are, essentially, a rescaling of lengths. A mass parameter introduces such a length scale.

On the contrary, the trace of the vacuum expectation value of the quantum energy-momentum tensor will – in the case of our massless scalar field – never be zero, even in the massless case. More precisely, the trace will formally vanish, but that is not a physically meaningful quantity. Individual components of the energy-momentum tensor are divergent – which may be understood by the existence of a infinite energy density in the ground state of fields – and these divergences must be dealt with using the formalism of regularization and renormalization. In this process, $\langle T^\mu_\mu \rangle$ acquires a trace.

Because the resulting theory is no longer conformally invariant – the now finite trace will be rescaled by the conformal factor – one speaks of the breaking of conformal invariance or symmetry breaking. Thus the phenomenon at hand is known as the conformal anomaly.

## 5.3 Two-dimensional gravitational action & dilaton gravity

Let's now assume spherical symmetry of the geometrical as well as the matter part of the action. Then no physical quantity depends upon the angular variables, and we can get rid of them in some fashion. The following section investigates how to reduce the dimension of spacetime from four to two.

If we want to set up a two-dimensional spacetime modeling a four-dimensional one and containing a matter field, we must consider two things: what kind of two-dimensional gravitation, i.e., Einstein equations to use, and what kind of matter in two dimensions.

### 5.3.1 The naïve reduction: ignoring two dimensions

The simplest way to get rid of the two angular dimensions is to ignore them. This would result in a two-dimensional gravitational action

$$S_\mathrm{g}^{(2)} = \frac{1}{16\pi} \int \mathrm{d}^2 x \, \sqrt{-g^{(2)}} \, R^{(2)} \ .$$

Even though one can, of course, consider such a model, it does not reflect the four-dimensional situation. Ignoring some dimensions inevitably leads to results that are qualitatively very different from those obtained for the original number of dimensions.

A simple example is a system emitting radiation, e.g. an oscillating electromagnetic dipole. Due to the conservation equations, the energy density of the radiation must be proportional to some decreasing function of the distance $r$ from the source. The dimension $n$ of the spacetime considered enters this function as a parameter. Speaking in an intuitive way, the power radiated off must disperse over an $(n-2)$-dimensional sphere of radius $r$. This means the energy density of radiation emitted from an antenna in our world decreases as $r^{-2}$ whereas in a two-dimensional spacetime, it doesn't decrease at all.

### 5.3.2 The more physical case: dilaton gravity

We do not run into the difficulties just mentioned if we do not alter the quantities we consider. That is, we propose an action $S^{(2)}$ which is physically related to $S^{(4)}$ instead of imitating it as far as the two common spacetime dimensions are concerned.

To achieve this, we perform all integrations over the angular variables, thereby getting rid of them and ending up not with different physics but with the two-dimensional aspects of physics in four dimensions. Spherical symmetry allows us to perform the integrations without difficulty.

A four-dimensional spherically symmetric metric is given by

$$\mathrm{d}s^2 = g^{(2)}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + r^2 \mathrm{d}\Omega^2$$

where $\alpha, \beta \in \{0, 1\}$. We generalize this metric by introducing the so-called dilaton field $\phi$,

$$\mathrm{d}s^2 = g^{(2)}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + e^{-2\phi(x)} \mathrm{d}\Omega^2 \ . \tag{5.13}$$

By putting $\phi = -\ln r$ we can always reconstruct the special case that corresponds to spherically symmetric four-dimensional spacetime.

For the determinant of the metric (5.13), we have

$$g^{(4)} = e^{-4\phi} \sin^2\theta \, g^{(2)} \ . \tag{5.14}$$

Using this relation in the Einstein-Hilbert action (5.3) and making use of the fact that $R^{(4)}$ does not depend upon $\theta$ and $\varphi$, we obtain

$$\begin{aligned} S^{(2)}_{\mathrm{g}} &= \frac{1}{16\pi} \int \mathrm{d}^2x \int_0^\pi \mathrm{d}\theta \, \sin\theta \int_0^{2\pi} \mathrm{d}\varphi \, \sqrt{-g^{(2)}} \, e^{-2\phi} R^{(4)} \\ &= \frac{1}{4} \int \mathrm{d}^2x \, \sqrt{-g^{(2)}} \, e^{-2\phi} R^{(4)} \ . \end{aligned} \tag{5.15}$$

If the metric components $g_{\alpha\beta}$ and hence the relation between $R^{(2)}$ and $R^{(4)}$ are known, one can obtain a two dimensional dilaton-dependent gravitational action from eqn. (5.15). The action thus obtained is suitable for meaningfully modeling the four-dimensional theory.

### 5.4 Matter in two dimensions and effective action

#### 5.4.1 The concept of effective action

As far as the conformal anomaly is concerned, the only property of the field the action $S$ depends upon is its spin. For the four-dimensional case, the anomaly and its spin dependence is given, e.g., in [4]. The conformal anomaly depends only on geometrical quantities [4]; thus, it may be used as a starting point for describing those aspects of the theory in question which likewise depend only on geometry. This leads to the so-called anomaly induced effective action.

An effective action contains only geometrical quantities, just as the trace anomaly. It is chosen in such a way that it leads to a vacuum expectation value of the energy-momentum tensor (by eqn. (5.7) or, more directly, eqn. (5.12)) whose trace coincides with the trace anomaly. This requirement determines only the anomaly induced part of the effective action; one can always add conformally invariant terms.

The effective action is used in order to calculate physical effects, among them the Hawking radiation. However, it turns out that the anomaly induced part alone is not sufficient, see below.

In the previous chapter, two possible geometries for two-dimensional spacetime models were discussed: an inherently two-dimensional spacetime, and a so-called dilaton model derived from four-dimensional spacetime by dimensional reduction under the assumption of spherical symmetry.

We will derive the effective action corresponding to a scalar field. For the genuine two-dimensional model, it is called the Polyakov action [15], for the more sophisticated model, it contains dilaton-dependent terms in addition to the Polyakov term.

#### 5.4.2 Genuinely two-dimensional matter and Polyakov action

Just as in the case of the gravitational action, the simplest two-dimensional model is one that shows no reminiscence of four dimensions. The two-dimensional pendant to the action (5.4) would then be

$$S_{\mathrm{m}}^{(2)} = -\frac{1}{4\pi} \int \mathrm{d}^2 x \sqrt{-g^{(2)}}\, (\nabla\Phi)^2 \ . \tag{5.16}$$

For this action, the trace anomaly reads

$$\langle T_\mu^\mu \rangle = \frac{R}{24\pi} \ . \tag{5.17}$$

*Derivation of the Polyakov action*

The Polyakov action can be obtained by integrating the trace anomaly as the energy-momentum tensor derives from the action through variation. In order to do this, we first introduce an auxiliary metric

$$g_{\mu\nu} = e^{-2\tau\sigma} \eta_{\mu\nu} \ , \qquad g^{\mu\nu} = e^{2\tau\sigma} \eta^{\mu\nu} \ .$$

In the end, we will put $\tau = 1$. Let, in the following equations, a bar mark quantities referring to the Minkowski metric $\eta_{\mu\nu}$. Then we have for the two-dimensional case considered here

$$\sqrt{-g} = e^{-2\tau\sigma} = e^{-2\tau\sigma}\sqrt{-\bar{g}} \qquad \text{and}$$

$$\Box = e^{2\tau\sigma}\partial_\mu e^{-2\tau\sigma}\left(e^{2\tau\sigma}\eta^{\mu\nu}\right)\partial_\nu = e^{2\tau\sigma}\bar{\Box}.$$

This leads immediately to the invariant quantity

$$\sqrt{-g}\,\Box = \sqrt{-\bar{g}}\,\bar{\Box}\ . \tag{5.18}$$

Now we consider the effective action in the model with parameter $\tau$; let's call it $S_\mathrm{e}(\tau)$. First of all, we take its derivative with respect to $\tau$ and make use of eqn. (5.7):

$$\begin{aligned}\frac{\mathrm{d}S_\mathrm{e}(\tau)}{\mathrm{d}\tau} &= \int \mathrm{d}^2x\,\frac{\delta S_\mathrm{e}(\tau)}{\delta g_{\mu\nu}}\frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}\tau} = -\int \mathrm{d}^2x\,\frac{\sqrt{-g}}{2}\langle T^{\mu\nu}\rangle\frac{\mathrm{d}\left(e^{-2\sigma\tau}\eta_{\mu\nu}\right)}{\mathrm{d}\tau}\\ &= \int \mathrm{d}^2x\,\sqrt{-g}\,\langle T^{\mu\nu}\rangle\cdot\sigma g_{\mu\nu} = \int \mathrm{d}^2x\,\sqrt{-g}\,\sigma\langle T^\mu_\mu\rangle\ .\end{aligned} \tag{5.19}$$

For the curvature scalar, we have (see, e.g., [7])

$$R = -2\tau\Box\sigma\ , \quad \sigma = -\frac{1}{2\tau}\frac{1}{\Box}R\ . \tag{5.20}$$

Plugging the trace anomaly (5.17) and eqn. (5.20) into eqn. (5.19) we obtain

$$\frac{\mathrm{d}S_\mathrm{e}(\tau)}{\mathrm{d}\tau} = \frac{1}{24\pi}\int \mathrm{d}^2x\,\sqrt{-g}\,\sigma R = -\frac{\tau}{12\pi}\int \mathrm{d}^2x\,\sqrt{-g}\,\sigma\Box\sigma\ . \tag{5.21}$$

Because of the invariance (5.18), the integral does not depend upon $\tau$.

Next, we integrate eqn. (5.21) with respect to $\tau$ from 0 to 1. The value $\tau = 0$ corresponds to Minkowski space. As Minkowski space is translation invariant, there is no coordinate dependence of $S_\mathrm{e}(0)$. With other variables being absent, it must be a constant which can, by way of renormalization, be put equal to zero. Therefore, we have

$$S_\mathrm{e}(1) = -\frac{1}{12\pi}\int_0^1 \mathrm{d}\tau\,\tau\int \mathrm{d}^2x\,\sqrt{-g}\,\sigma\Box\sigma = -\frac{1}{24\pi}\int \mathrm{d}^2x\,\sqrt{-g}\,\sigma\Box\sigma\ .$$

This is already the Polyakov action; let's rewrite it using eqn. (5.20) with $\tau = 1$:

$$S_\mathrm{Pol} = -\frac{1}{96\pi}\int \mathrm{d}^2x\,\sqrt{-g}R\frac{1}{\Box}R\ . \tag{5.22}$$

Of course, this quantity is only defined up to the dependence of the inverse d'Alembertian on the boundary conditions the scalar field has to satisfy.

*The energy-momentum tensor*

First of all, we write down the general variation of the Polyakov action (5.22):

$$\delta S_\mathrm{Pol} = -\frac{1}{96\pi}\int \mathrm{d}^2x\,\left\{\left(\delta\sqrt{-g}\right)R\frac{1}{\Box}R + 2\sqrt{-g}\,(\delta R)\frac{1}{\Box}R + \sqrt{-g}\,R\left(\delta\frac{1}{\Box}\right)R\right\}\ . \tag{5.23}$$

By eqn. (1.22) and (1.17) we obtain

$$\int \mathrm{d}^2x\,\sqrt{-g}\,R\left(\delta\frac{1}{\Box}\right)R = -\int \mathrm{d}^2x\,\sqrt{-g}\,R\frac{1}{\Box}(\delta\Box)\frac{1}{\Box}R = -\int \mathrm{d}^2x\,\sqrt{-g}\left[\frac{1}{\Box}R\right](\delta\Box)\frac{1}{\Box}R\ .$$

Now we plug this into eqn. (5.23) and perform the variations by eqn. (1.19), (1.20), and (1.22),

$$\delta S_{\mathrm{Pol}} = -\frac{1}{96\pi}\int \mathrm{d}^2x\,\sqrt{-g}\,(\delta g_{\mu\nu})\left\{ \frac{1}{2}g^{\mu\nu}R\frac{1}{\Box}R + 2\left(\left[\frac{1}{\Box}R\right]^{;\mu\nu} - g^{\mu\nu}R - R^{\mu\nu}\frac{1}{\Box}R\right)\right.$$
$$\left. -\left[\frac{1}{\Box}R\right]^{,(\mu}\left[\frac{1}{\Box}R\right]^{,\nu)} + \frac{1}{2}g^{\mu\nu}\left(\left[\frac{1}{\Box}R\right]^{,\varepsilon}\left[\frac{1}{\Box}R\right]_{,\varepsilon} + \left[\frac{1}{\Box}R\right]R\right)\right\} .$$

Here, marked terms cancel by the identity (2.2), leaving the expression:

$$\delta S_{\mathrm{Pol}} = -\frac{1}{96\pi}\int \mathrm{d}^2x\,\sqrt{-g}\,(\delta g_{\mu\nu})\left\{2\left[\frac{1}{\Box}R\right]^{;\mu\nu} - \left[\frac{1}{\Box}R\right]^{,\mu}\left[\frac{1}{\Box}R\right]^{,\nu}\right.$$
$$\left. -g^{\mu\nu}\left(-\frac{1}{2}\left[\frac{1}{\Box}R\right]^{,\varepsilon}\left[\frac{1}{\Box}R\right]_{,\varepsilon} + 2R\right)\right\} . \tag{5.24}$$

We can now read off from (5.24) the vacuum expectation value of the energy-momentum tensor arising from $S_{\mathrm{Pol}}$ according to (5.7),

$$\langle T^{\mu\nu}_{\mathrm{Pol}}\rangle = \frac{1}{48\pi}\left\{2\left[\frac{1}{\Box}R\right]^{;\mu\nu} - \left[\frac{1}{\Box}R\right]^{,\mu}\left[\frac{1}{\Box}R\right]^{,\nu} - g^{\mu\nu}\left(2R - \frac{1}{2}\left[\frac{1}{\Box}R\right]^{,\varepsilon}\left[\frac{1}{\Box}R\right]_{,\varepsilon}\right)\right\} . \tag{5.25}$$

This energy-momentum tensor agrees with [7] up to an overall factor of $-2$ which would achieve consistency of the cited paper. This factor is present in the later publication [14].

We can simplify the expression (5.25) by writing the curvature scalar $R$ in terms of the conformal factor,

$$R = -2\Box\sigma\ , \quad \sigma = -\frac{1}{2}\frac{1}{\Box}R\ ,$$

and choosing null coordinates (2.4). This yields

$$\langle T^{\mu\nu}_{\mathrm{Pol}}\rangle = -\frac{1}{12\pi}\left(\sigma^{;\mu\nu} + \sigma^{,\mu}\sigma^{,\nu}\right) = -\frac{1}{12\pi}\big((\partial^\mu\sigma)(\partial^\nu\sigma) - \partial_\mu\partial_\nu\sigma\big)$$

which can be proven to satisfy the conservation equations.

*Hawking radiation and comparison with the four-dimensional theory*

We want to shortly summarize the discussion of this result by R. Balbinot and A. Fabbri [7].

The authors evaluate the expectation value for three specific vacua:

1. Boulware vacuum $|B\rangle$. This state is defined as the asymptotic vacuum, i.e., there are no particles far away from the Black Hole. Field modes are not regular on the event horizon; suitable ones are those with respect to Schwarzschild coordinates $t, r$.

   In the Polyakov model, these properties are retained. The components of the energy-momentum tensor vanish at infinity.

2. Hartle-Hawking vacuum $|H\rangle$. In this state, the radiating Black Hole can be regarded as being in equilibrium with a thermal bath of particles outside or, equivalently, as being surrounded by a reflecting shell. Field modes are regular on the horizon which

is related to the asymptotic behaviour of $\langle T_{\mu\nu}\rangle$ near $r = 2M$. (These relations go back to Christensen and Fulling [16].) Field modes are defined with respect to Kruskal coordinates.

In the two-dimensional model, the same properties hold, and the state $|H\rangle$ defined there is a thermal state at the Hawking temperature $T_H = 1/(8\pi M)$.

3. Unruh vacuum $|U\rangle$. The Unruh state corresponds to the gravitational collapse of spherically symmetric matter to form a Black Hole, see chapter 4. Field modes are required to be regular on the future event horizon.

   Again, the Polyakov model is able to reproduce the features of four-dimensional gravitational collapse as far as Hawking radiation is concerned. It leads to thermal radiation at the correct Hawking temperature.

Thus, the combination of this genuine two-dimensional matter and the dilaton-dependent gravitational action (5.15) yields a surprisingly good qualitative agreement with the four-dimensional setting with respect to Hawking radiation.

However, this combination is questionable as it is rather inconsistent: two different schemes of dimensional reduction have been used in order to obtain the gravitational and matter parts of the action. Therefore, it would be desirable to use a concept of matter in two dimensions which is rooted in the four-dimensional one as well.

### 5.4.3 Spherically symmetric matter in four dimensions

*The matter action in the dilaton model*

Making use of spherical symmetry and the relation (5.14), we can obtain a matter action for the two-dimensional dilaton model from the four-dimensional matter action (5.4),

$$\begin{aligned} S_{\mathrm{m}}^{(2)} &= -\frac{1}{(4\pi)^2}\int \mathrm{d}^2x \int_0^\pi \mathrm{d}\theta\, \sin\theta \int_0^{2\pi} \mathrm{d}\varphi\, \sqrt{-g^{(2)}}\, e^{-2\phi}(\nabla\Phi)^2 \\ &= -\frac{1}{4\pi}\int \mathrm{d}^2x\, \sqrt{-g^{(2)}}\, e^{-2\phi}(\nabla\Phi)^2 \ . \end{aligned} \tag{5.26}$$

Just as in the case of the gravitational action, this matter action does not imply physics different from the four-dimensional one but rather describes only that aspect of four-dimensional physics which is spherically symmetric. This is also known as the s-channel, and the approximation made by requiring spherical symmetry is called the s-wave approximation.

As roughly 90% of the Hawking flux are contributed by the s-channel, this approximation makes rather good sense.

*Trace anomaly and effective action*

We just derived the effective action for a massless scalar field in genuine two-dimensional spacetime. Now we include the dilaton field $\phi$.

For a dilaton model of a scalar field theory with the action (5.26), the trace anomaly reads [6, 17, 7]

$$\langle T^\mu_\mu\rangle = \frac{1}{24\pi}\left(R - 6(\nabla\phi)^2 + 6\Box\phi\right) \ . \tag{5.27}$$

There has been a controversy about this expression. It is the subject of the following subsection.

By a derivation similar to that of the Polyakov action, we can obtain the anomaly induced effective action for our dimensionally reduced dilaton model [7],

$$S_{\mathrm{aind}} = -\frac{1}{96\pi}\int \mathrm{d}^2x\,\sqrt{-g}\left(R\frac{1}{\Box}R - 12(\nabla\phi)^2\frac{1}{\Box}R + 12\phi R\right)\ . \tag{5.28}$$

*The energy-momentum tensor*

We start by splitting the anomaly-induced effective action (5.28) into the Polyakov action (5.22) and the dilaton dependent part,

$$S_{\mathrm{aind}} = S_{\mathrm{Pol}} - \frac{1}{8\pi}\int \mathrm{d}^2x\,\sqrt{-g}\left[\phi R - (\nabla\phi)^2\frac{1}{\Box}R\right]\ . \tag{5.29}$$

Now we perform the variation; in the following equations, marked terms cancel according to (2.2):

$$\begin{aligned}
\delta S_{\mathrm{aind}} &= \delta S_{\mathrm{Pol}} - \frac{1}{8\pi}\int \mathrm{d}^2x\,\left\{\left(\delta\sqrt{-g}\right)\left[\phi R - (\nabla\phi)^2\frac{1}{\Box}R\right] + \sqrt{-g}\,\delta\left[\phi R - (\nabla\phi)^2\frac{1}{\Box}R\right]\right\}\\
&= \delta S_{\mathrm{Pol}} - \frac{1}{8\pi}\int \mathrm{d}^2x\,\sqrt{-g}\left\{\frac{1}{2}g^{\mu\nu}(\delta g_{\mu\nu})\left[\phi R - (\nabla\phi)^2\frac{1}{\Box}R\right] + \phi(\delta R)\right.\\
&\qquad\qquad \left. - \left(\delta g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\right)\frac{1}{\Box}R - (\nabla\phi)^2\left(\delta\frac{1}{\Box}\right)R - (\nabla\phi)^2\frac{1}{\Box}(\delta R)\right\}\\
&= \delta S_{\mathrm{Pol}} - \frac{1}{8\pi}\int \mathrm{d}^2x\,\sqrt{-g}\left\{(\delta g_{\mu\nu})\left(\frac{1}{2}g^{\mu\nu}\left[\phi R - (\nabla\phi)^2\frac{1}{\Box}R\right] + \phi^{;\mu\nu} - g^{\mu\nu}\Box\phi - R^{\mu\nu}\phi\right.\right.\\
&\qquad\qquad \left.\left. + \phi^{,\mu}\phi^{,\nu}\frac{1}{\Box}R\right) + \left[\frac{1}{\Box}(\nabla\phi)^2\right](\delta\Box)\frac{1}{\Box}R - \left[\frac{1}{\Box}(\nabla\phi)^2\right](\delta R)\right\}\\
&= \delta S_{\mathrm{Pol}} - \frac{1}{8\pi}\int \mathrm{d}^2x\,\sqrt{-g}\,(\delta g_{\mu\nu})\left\{-\frac{1}{2}g^{\mu\nu}(\nabla\phi)^2\frac{1}{\Box}R + \phi^{;\mu\nu} - g^{\mu\nu}\Box\phi + \phi^{,\mu}\phi^{,\nu}\frac{1}{\Box}R\right.\\
&\qquad + \left[\frac{1}{\Box}(\nabla\phi)^2\right]^{,(\mu}\left[\frac{1}{\Box}R\right]^{,\nu)} - \frac{1}{2}g^{\mu\nu}\left(\left[\frac{1}{\Box}(\nabla\phi)^2\right]^{,\varepsilon}\left[\frac{1}{\Box}R\right]_{,\varepsilon} + \left[\frac{1}{\Box}(\nabla\phi)^2\right]R\right)\\
&\qquad\qquad \left. - \left[\frac{1}{\Box}(\nabla\phi)^2\right]^{;\mu\nu} + g^{\mu\nu}(\nabla\phi)^2 + R^{\mu\nu}\frac{1}{\Box}(\nabla\phi)^2\right\}\ .
\end{aligned} \tag{5.30}$$

We can now read off from eqn. (5.30) the vacuum expectation value of the energy-momentum tensor according to (5.7) where $\langle T^{\mu\nu}_{\mathrm{Pol}}\rangle$ is given by (5.25),

$$\begin{aligned}
\langle T^{\mu\nu}_{\mathrm{aind}}\rangle = \langle T^{\mu\nu}_{\mathrm{Pol}}\rangle + \frac{1}{4\pi}&\left\{\phi^{,\mu}\phi^{,\nu}\frac{1}{\Box}R + \left[\frac{1}{\Box}(\nabla\phi)^2\right]^{,(\mu}\left[\frac{1}{\Box}R\right]^{,\nu)} - \left[\frac{1}{\Box}(\nabla\phi)^2\right]^{;\mu\nu} + \phi^{;\mu\nu}\right.\\
&\left. -\frac{1}{2}g^{\mu\nu}\left((\nabla\phi)^2\frac{1}{\Box}R + \left[\frac{1}{\Box}(\nabla\phi)^2\right]^{,\varepsilon}\left[\frac{1}{\Box}R\right]_{,\varepsilon} - 2(\nabla\phi)^2 + 2\Box\phi\right)\right\}\ .
\end{aligned}$$

Again, this agrees with the result given in [7] up to a factor of $-2$ which is fixed in [14].

Again we refer to the discussion in [7]. As pointed out by the cited authors, the Boulware state $|B\rangle$ corresponding to asymptotic Minkowski vacuum is still properly described by the two-dimensional model. However, the model yields a negative Hawking flux ($(u, u)$ and $(v, v)$ components of the energy-momentum tensor) for the Hartle-Hawking state $|H\rangle$.

Therefore, one wishes to improve the model. One requirement $S_{\text{aind}}$ had to fulfill was that it should yield an energy-momentum tensor whose trace is just the conformal anomaly. By adding conformally invariant parts to the action, one gets, by eqn. (5.12), only trace-free contributions to the energy-momentum tensor. Thus one is free to do so without violating the trace condition on the effective action.

One such conformally invariant contribution to $S_{\text{aind}}$ has been proposed by V. Mukhanov, A. Wipf, and A. Zelnikov [6]. They obtain the correction to $S_{\text{aind}}$ from a classical approximation of the heat kernel. The problem of negative Hawking flux in the state $|H\rangle$ can be cured by their approach. However, the properties of the Minkowski vacuum are changed as well, to the effect that Minkowski vacuum is not a solution of the field equations derived from the altered action. Therefore, the contribution to the effective action proposed by Mukhanov et.al. is not satisfactory.

Another suggestion is due to A. Balbinot and R. Fabbri [7]. They start from a couple of physical requirements the two-dimensional energy-momentum tensor ought to fulfill. These are the conservation equations, the vanishing of the tensor components for Minkowski vacuum, and state independence of the trace of the energy-momentum tensor. These requirements reflect the properties of the pendant four-dimensional energy-momentum tensor.

The correction actually made is, however, only defined in an ad hoc way. Though it can qualitatively reproduce the behaviour of the four-dimensional energy-momentum tensor in all three states $|B\rangle$, $|H\rangle$, and $|u\rangle$, it would be preferrable to be able to derive the correction from the knowledge of the four-dimensional theory.

This has been done by Y. Gusev and A. Zelnikov [8]. Their proposal will be the subject of the following chapter.

### 5.4.4 The controversy about the anomaly induced effective action

There had been a discussion about the trace anomaly and induced effective action in the dilaton case. The subject of controversy was the coefficient in front of the $\Box\phi$ and $\phi R$ term, resp.

The apparent ambiguity of this coefficient was an open question when this diploma work began; it was part of the subject of this work to consider this question.

In the course of this work I convinced myself that a recent paper by J.S. Dowker [5] actually contains the solution to the puzzle; by then, this result had been generally accepted by the colleagues. I will explain the different points of view and how they can be reconciled in this section.

*The point of controversy*

The trace anomaly of the dilaton scalar field theory, eqn. (5.27), has been given in several publications, among them [18, 6, 17, 19, 20, 21]. They all agree on

$$\langle T^\mu_\mu \rangle = \frac{1}{24\pi}\left(R - 6(\nabla\phi)^2 + \alpha\Box\phi\right)$$

where $\alpha$ is the coefficient in question [5]. The corresponding anomaly induced effective action reads

$$S_{\text{aind}} = -\frac{1}{96\pi}\int \mathrm{d}^2x\,\sqrt{-g}\left(R\frac{1}{\Box}R - 12(\nabla\phi)^2\frac{1}{\Box}R + 2\alpha\phi R\right) \, .$$

For the coefficient $\alpha$ there exist, however, three suggestions.

1. $\alpha = 6$ obtained by Elizalde et.al. [18] and V. Mukhanov, A. Wipf, and A. Zelnikov [6]. This result turned out to be correct in the end and is thus used in this work.
2. $\alpha = -2$ proposed by R. Bousso and S.W. Hawking [20] which turned out to be a mistake.
3. $\alpha = 4$ obtained by Kummer et.al. [17, 19, 21] for the same setup of the two-dimensional model as was used by Bousso and Hawking.

In order to sort out this variety of apparently contradictory results, it is not necessary to start the calculations all over again, repeating the procedure of regularization and renormalization. It is enough to have a closer look at the cited publications in order to see what statements are really made and where real mistakes occur. This is what Dowker did [5].

*The value $\alpha = -2$: a mistake*

The only result which involves a real mathematical mistake is that obtained by Bousso and Hawking.

If we derive the effective action (5.28) from the trace anomaly (5.27) by a procedure similar to our derivation of the Polyakov action above, we get as an intermediate result the expression

$$S_{\text{aind}} = -\frac{1}{96\pi}\int \mathrm{d}^2x\,\sqrt{-g}\left(R\frac{1}{\Box}R - 12(\nabla\phi)^2 R + 12\Box\phi\frac{1}{\Box}R\right) \, .$$

The last term is the one with the coefficient in question. In order to turn it into an expression of the form $\phi R$, we have to account for the zero modes of the d'Alembertian as the inverse d'Alembertian is defined up a linear combination of these zero modes.

We will not concern ourselves with the exact nature of these zero modes. All we need to know is that they are different for different topologies of the two-dimensional spacetime under consideration. In particular, a spacetime with the topology of a two-dimensional plane has different zero modes than one with the topology of a two-sphere [5].

In [20], Bousso and Hawking do not consistently work with any one of these topologies but rather switch from a spacetime with plane topology to one with a two-sphere topology without paying attention to the extra zero modes. This is what causes their calculation to fail.

*The generalized spacetime model by Kummer et.al.*

In [17], Kummer and his co-authors introduce a more general model of dilaton gravity than we have used so far. Instead of the dilaton field, they use different functions of the dilaton in the matter and gravitational parts of the action. They call these functions $\varphi(\phi)$ in the matter part and $\psi(\phi)$ in the gravitational part.

From their point of view, the dilaton gravity we have considered so far is just the special case where these functions are the same. (And of this special case, the dimensionally reduced four-dimensional theory is, in turn, just a special case.)

It turns out that the effective action obtained from this more general model in the case $\varphi(\phi) = \psi(\phi) = \phi$ is exactly the same as that obtained in, e.g., [6], i.e., $\alpha = 6$. This is just a check of both calculations as this choice of $\varphi$ and $\psi$ describes the model used in [6], and thus agreement was to be expected.

In the calculation examined above, Bousso and Hawking use a model where $\varphi(\phi) = \phi$ and $\psi(\phi) = 0$. This assumes a matter action as we have considered all along but a gravitational action that ignores the dilaton. If we consider this model and calculate the effective action without making the same mistake as Bousso and Hawking, we find $\alpha = 4$, the result also obtained for this model by Ichinose [21].

Thus there is not any contradiction between the $\alpha = 4$ and $\alpha = 6$ results. People have simply considered different models of dilaton gravity. For the model used throughout this work, $\alpha = 6$ is a certain fact.

Chapter 6

# A Conformally Invariant Correction to the Effective Action

In the previous chapter, we derived an effective action for our two-dimensional dilaton model. This effective action is determined uniquely by the conformal anomaly arising from the scalar quantum field. Like that anomaly, it contains only geometrical quantities: the curvature scalar, the dilaton, and, implicitly, the metric via the covariant derivatives.

However, trying to reconstruct Hawking radiation from it, we ended up with a qualitative difference from the well-known four-dimensional result.

In order to cure this shortcoming, people have tried to add conformally invariant terms to the effective action or, equivalently, suitable contributions to the energy momentum tensor. These do not change the resulting trace of the energy momentum tensor and thus are not in conflict with the conformal anomaly, but yield contributions to the Hawking flux which – hopefully – result in a qualitative agreement between the two-dimensional and the four-dimensional theory.

One such attempt has been made by R. Balbinot and A. Fabbri [7]. These authors propose a contribution to the energy momentum tensor determined by some physical requirements.

An alternative contribution to the effective action has been suggested by Y. Gusev and A. Zelnikov [8]. Instead of an educated guess, they give a strict derivation using heat kernel regularization. It is this latter proposal we shall work on in this chapter.

## 6.1 The contribution to the effective action

### 6.1.1 The proposal made by Gusev and Zelnikov

In [8], an improved effective action for the two-dimensional dilaton model is given, namely

$$S_{\text{eff}} = S_{\text{aind}} + S_{(2)} + S_{(3)} \ ,$$

$$S_{(2)} = -\frac{1}{8\pi}\int \mathrm{d}^2x\,\sqrt{-g}\,(\Box\phi)\ln\Big(\frac{-\Box}{\mu^2}\Big)\phi \ , \tag{6.1}$$

$$S_{(3)} = \frac{1}{8\pi}\int \mathrm{d}^2x\,\sqrt{-g}\,\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}\frac{1}{\Box_1}R_1(\Box\phi_2)(\Box\phi_3) \ . \tag{6.2}$$

The anomaly-induced effective action, $S_{\text{aind}}$, is given by eqn. (5.29). $S_{(2)}$ and $S_{(3)}$ are the conformally invariant corrections to second and third order in curvatures, resp. The cited authors work, however, in the Euclidean section with metric (++), so we use different signs than they do. In (6.1) and (6.2) the following definitions have been used [8]:

$$\ln\Big(\frac{-\Box}{\mu^2}\Big) \equiv -\int\limits_0^\infty \mathrm{d}m^2 \left(\frac{1}{m^2-\Box} - \frac{1}{m^2-\mu^2}\right) \ , \tag{6.3}$$

$$\frac{\ln(\Box_1/\Box_2)}{\Box_1-\Box_2} \equiv -\int\limits_0^\infty \mathrm{d}m^2 \, \frac{1}{m^2-\Box_1}\frac{1}{m^2-\Box_2} \ . \tag{6.4}$$

In (6.3) the regularization parameter $\mu$ occurs; this notation is, however, only symbolic. Later in this chapter, we will see explicitly how the regularization is done. Indices 1 and 2 in (6.4) indicate points in spacetime where the d'Alembertian in question acts.

The derivation of the corrections $S_{(2)}$ and $S_{(3)}$ in [8] already fixes the boundary conditions: They describe the spherically symmetric collapse of a cloud of dust in the distant past. 'Distant' means that we have approximately stationary radiation at the time when we evaluate the energy momentum tensor. The scalar field had been in the vacuum state of the asymptotic Minkowski space that existed long before the collapse.

In order to restrict considerations to the physical case, in other words, in order to ensure causality, the Green function implicit in the inverse differential operators (see eqn. (1.15)) must be the retarded one which will be calculated below.

In the next sections, we will derive the contributions to the energy momentum tensor that arise from $S_{(2)}$ and $S_{(3)}$. These results have not been previously published.

### 6.1.2 The retarded Green function

In the following, we will need the retarded, massive Green function $G^m_{\mathrm{R}}(x,y)$. It is supported only where $y$ is in the causal past of $x$ which is the physical case.

Up to zeroth order in curvatures, $G^m_{\mathrm{R}}$ is the same for conformally flat space as for Minkowski space [22]; let's now calculate $G^m_{\mathrm{R}}$ for Minkowski space.

We know the Green function in momentum representation and can obtain the position representation by a Fourier transform. We will calculate $G^m_{\mathrm{R}}$ via Wightman functions [23].

Within this subsection, we will write $t$, $r$ instead of $x^t-y^t$, $x^r-y^r$.

Wightman functions are defined by

$$\begin{aligned} G^m_\pm(x,y) &= \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\mathrm{d}k^1\, e^{ik^1r}\int\limits_{\mathscr{C}_\pm}\mathrm{d}k^0\, e^{-ik^0t}\frac{1}{-(k^0)^2+\omega_k^2} \ , \\ \omega_k^2 &= (k^1)^2+m^2 \ . \end{aligned}$$

The contours of integration are closed loops running clockwise around the singularities in the complex $k^0$ plane. $\mathscr{C}_\pm$ encircles $k^0=\pm\omega_k$,

$$\begin{aligned} G^m_\pm(x,y) &= \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\mathrm{d}k^1\, e^{ik^1r}\int\limits_{\mathscr{C}_\pm}\mathrm{d}k^0\, e^{-ik^0t}\frac{1}{2\omega_k}\left(\frac{1}{\omega_k+k^0}+\frac{1}{\omega_k-k^0}\right) \\ &= \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\mathrm{d}k^1\, e^{ik^1r}\frac{1}{2\omega_k}(-2\pi i)\operatorname*{Res}_{k^0=\pm\omega_k}\frac{e^{-ik^0t}}{\omega_k\mp k^0} \ . \end{aligned} \tag{6.5}$$

We calculate the residua at the singularities $k^0 = \pm\omega_k$,

$$\frac{e^{-ik^0 t}}{\omega_k \mp k^0} = \mp e^{\mp i\omega_k t} \frac{e^{-i(k^0 \mp \omega_k)t}}{k^0 \mp \omega_k} = \mp e^{\mp i\omega_k t} \frac{1}{k^0 \mp \omega_k} \Big[ 1 - it(k^0 \mp \omega_k) + \mathscr{O}(k^0 \mp \omega_k) \Big] ,$$

$$\text{thus} \quad \operatorname*{Res}_{k^0 = \pm\omega_k} \frac{e^{-ik^0 t}}{\omega_k \mp k^0} = \mp e^{\mp i\omega_k t} ,$$

and plug them into (6.5),

$$G^m_\pm(x,y) = \frac{\pm i}{4\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}k^1 \, e^{ik^1 r} \frac{1}{\omega_k} e^{\mp i\omega_k t} . \tag{6.6}$$

Now we introduce a parameter $\eta$ defined by

$$k^1 = m \sinh\eta , \quad \omega_k = m\cosh\eta ; \quad \mathrm{d}k^1 = \omega_k \, \mathrm{d}\eta .$$

Equation (6.6) now reads

$$G^m_\pm(x,y) = \frac{\pm i}{4\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}\eta \, \exp\big[ i(\mp tm\cosh\eta + rm\sinh\eta) \big] . \tag{6.7}$$

In the end, only the region inside the lightcone, $|t| \geq |r|$, is of interest. Then,

$$t\cosh\eta \pm r\sinh\eta = \operatorname{sgn}(t)\sqrt{t^2 - r^2} \, \cosh\left( \eta \pm \ln \tfrac{t+r}{t-r} \right) . \tag{6.8}$$

We now need to explicitely cut off the region outside the lightcone from Wightman functions using a step function as defined by eqn. (1.13). We insert (6.8) into (6.7), shift the integral appropriately, and make use of the fact that $\cosh\eta$ and $\cos\eta$ are even functions whereas $\sin\eta$ is odd. With $\sigma_{xy}$ given by eqn. (1.3), we obtain

$$\begin{aligned}
G^m_\pm(x,y) &= \frac{\pm i}{4\pi} \Theta(-\sigma_{xy}) \int\limits_{-\infty}^{\infty} \mathrm{d}\eta \, \exp\Big[ \mp i \operatorname{sgn}(t) m \sqrt{-2\sigma_{xy}} \, \cosh\eta \Big] \\
&= \frac{\pm i}{2\pi} \Theta(-\sigma_{xy}) \int\limits_0^{\infty} \mathrm{d}\eta \, \cos\left( m\sqrt{-2\sigma_{xy}} \, \cosh\eta \right) \\
&\quad \mp \frac{\operatorname{sgn}(t)}{2\pi} \Theta(-\sigma_{xy}) \int\limits_0^{\infty} \mathrm{d}\eta \, \sin\left( m\sqrt{-2\sigma_{xy}} \, \cosh\eta \right) \\
&= \Theta(-\sigma_{xy}) \left[ \frac{\mp i}{4} N_0\left( m\sqrt{-2\sigma_{xy}} \right) + \frac{\operatorname{sgn}(t)}{4} J_0\left( m\sqrt{-2\sigma_{xy}} \right) \right]
\end{aligned} \tag{6.9}$$

where $N_0$ and $J_0$ denote the Neumann and Bessel function of index 0, resp.

The retarded Green function is defined in terms of Wightman functions by

$$G^m_{\mathrm{R}}(x,y) = \Theta(t) \left[ G^m_+(x,y) + G^m_-(x,y) \right] .$$

We insert (6.9) and obtain the massive retarded Green function,

$$G_{\mathrm{R}}^{m}(x,y) = \frac{1}{2}\Theta(x^t - y^t)\Theta(-\sigma_{xy})J_0\left(m\sqrt{-2\sigma_{xy}}\right) \ . \tag{6.10}$$

The wave equation satisfied is

$$(\Box_x - m^2)G_{\mathrm{R}}^{m}(x,y) = -\delta(x-y) \ ,$$

and the inverse wave operator acting on a function $\Psi$ reads

$$\left[\frac{1}{m^2-\Box}\Psi\right](x) = \int \mathrm{d}^2y\,\sqrt{-g(y)}\,G_{\mathrm{R}}^{m}(x,y)\Psi(y) \ . \tag{6.11}$$

$G_{\mathrm{R}}^{m}(x,y)$ depends only on differences of coordinates, therefore

$$\frac{\partial}{\partial x^\mu}G_{\mathrm{R}}^{m}(x,y) = -\frac{\partial}{\partial y^\mu}G_{\mathrm{R}}^{m}(x,y) \ .$$

## 6.2 The second order correction

### 6.2.1 Variation

Analogously to the procedure in the previous chapter, we have to vary the correction to the effective action with respect to the metric in order to determine its contribution to the energy momentum tensor.

The general form of the variation of (6.1) reads

$$\delta S_{(2)} = \frac{-1}{8\pi}\int \mathrm{d}^2x\,\Big\{(\delta\sqrt{-g})(\Box\phi)\ln\Big(\frac{-\Box}{\mu^2}\Big)\phi + \sqrt{-g}\,\left[(\delta\Box)\phi\right]\ln\Big(\frac{-\Box}{\mu^2}\Big)\phi$$
$$+\sqrt{-g}\,(\Box\phi)\delta\ln\Big(\frac{-\Box}{\mu^2}\Big)\phi\Big\} \ . \tag{6.12}$$

When we apply (1.22) to (6.3), the term $\dfrac{1}{m^2-\Box}$ does not contribute as it doesn't depend on the metric:

$$\delta\ln\Big(\frac{-\Box}{\mu^2}\Big) = -\int_0^\infty \mathrm{d}m^2\,\delta\frac{1}{m^2-\Box} = -\int_0^\infty \mathrm{d}m^2\,\frac{1}{m^2-\Box}(\delta\Box)\frac{1}{m^2-\Box} \ .$$

Now we use this in the last term of (6.12) and then apply (1.17):

$$\int \mathrm{d}^2x\,\sqrt{-g}\,(\Box\phi)\,\delta\ln\Big(\frac{-\Box}{\mu^2}\Big)\phi = -\int \mathrm{d}^2x\,\sqrt{-g}\int_0^\infty \mathrm{d}m^2\,(\Box\phi)\frac{1}{m^2-\Box}(\delta\Box)\frac{1}{m^2-\Box}\phi$$
$$= -\int \mathrm{d}^2x\,\sqrt{-g}\int_0^\infty \mathrm{d}m^2\,\Big[\frac{1}{m^2-\Box}\Box\phi\Big](\delta\Box)\frac{1}{m^2-\Box}\phi \ .$$

At last, we plug this result into (6.12) and perform the variations by (1.19) and (1.21), resp.:

$$\begin{aligned}\delta S_{(2)} = \frac{-1}{8\pi}\int \mathrm{d}^2x\,\sqrt{-g}\,(\delta g_{\mu\nu})\Bigg\{ &\frac{1}{2}g^{\mu\nu}(\Box\phi)\ln\Big(\frac{-\Box}{\mu^2}\Big)\phi \\ &+\left[\ln\Big(\frac{-\Box}{\mu^2}\Big)\phi\right]^{,(\mu}\phi^{,\nu)} - \frac{1}{2}g^{\mu\nu}\left(\left[\ln\Big(\frac{-\Box}{\mu^2}\Big)\phi\right]^{,\varepsilon}\phi_{,\varepsilon} + \left[\ln\Big(\frac{-\Box}{\mu^2}\Big)\phi\right]\Box\phi\right) \\ -\int_0^\infty \mathrm{d}m^2\,\Bigg[\!\!\Bigg[&\left[\frac{1}{m^2-\Box}\Box\phi\right]^{,(\mu}\left[\frac{1}{m^2-\Box}\phi\right]^{,\nu)} \\ -\frac{1}{2}g^{\mu\nu}&\left(\left[\frac{1}{m^2-\Box}\Box\phi\right]^{,\varepsilon}\left[\frac{1}{m^2-\Box}\phi\right]_{,\varepsilon} + \left[\frac{1}{m^2-\Box}\Box\phi\right]\Box\frac{1}{m^2-\Box}\phi\right)\Bigg]\!\!\Bigg]\Bigg\}\ . \end{aligned} \tag{6.13}$$

### 6.2.2 The stress tensor

The freedom to choose a particular coordinate system allows us to make the expression (6.13) easier to handle. Light cone coordinates with the metric (5.2) simplify the diagonal stress tensor components considerably: only terms without $g^{\mu\nu}$ survive.

Of the $(u,u)$ and $(v,v)$ components, we consider $\langle T^{uu}_{(2)}\rangle$. The other one has the same form, only with $v$ in place of $u$. Keeping (1.18) in mind, we can almost immediately read off the $(u,u)$ component from (6.13),

$$\langle T_{(2)uu}\rangle = \frac{1}{4\pi}\left\{\left[\ln\Big(\frac{-\Box}{\mu^2}\Big)\phi\right]_{,u}\phi_{,u} - \int_0^\infty \mathrm{d}m^2\left[\frac{1}{m^2-\Box}\Box\phi\right]_{,u}\left[\frac{1}{m^2-\Box}\phi\right]_{,u}\right\}\ .$$

By eqn. (6.3), this can be written as

$$\langle T_{(2)uu}\rangle = \frac{1}{4\pi}\int_0^\infty \mathrm{d}m^2\left\{\frac{1}{m^2-\Box}\phi_{,u}\phi_{,u} - \left[\frac{1}{m^2-\Box}\Box\phi\right]_{,u}\left[\frac{1}{m^2-\Box}\phi\right]_{,u}\right\}\ .$$

The inverse wave operators can then be expressed in terms of the retarded Green function by eqn. (6.11) where the Green function itself is given by eqn. (6.10),

$$\begin{aligned}\langle T_{(2)uu}\rangle = \frac{1}{4\pi}\int_0^\infty \mathrm{d}m^2\Bigg\{&\phi_{,u}(x)\int \mathrm{d}^2y\,\sqrt{-g(y)}\,G^m_{\mathrm{R}}(x,y)\phi_{,u}(y) \\ -&\left[\int \mathrm{d}^2y\,\sqrt{-g(y)}\,G^m_{\mathrm{R}}(x,y)\Box_y\phi(y)\right]_{,u}\left[\int \mathrm{d}^2z\,\sqrt{-g(z)}\,G^m_{\mathrm{R}}(x,z)\phi(z)\right]_{,u}\Bigg\}\ . \end{aligned} \tag{6.14}$$

If we want to evaluate this expression in the Schwarzschild geometry, we must use coordinates which can be conformally transformed into Minkowski coordinates. This is not possible with Schwarzschild coordinates; we will have to use the tortoise coordinate $r^*$ (see eqn. (2.5)) instead of the Schwarzschild radial coordinate $r$.

Thus we have to perform integrations with respect to the tortoise coordinate whereas the integrands depend on the Schwarzschild coordinate through the dilaton field, see eqn. (5.13) and the discussion thereafter. It seems that such an integral is not exactly solvable and only numerical results can be expected. The numerical evaluation of the expression (6.14) with (6.10) inserted for $G^m_{\mathrm{R}}$ is left for future work.

## 6.3 The third order correction

The conformally invariant correction to third order in the curvatures, $S_{(3)}$, is somewhat harder to handle. The particular difficulty arises from the fact that because of the non-locality – which results in quite a lot of integrations – sometimes up to five different spacetime points – integration variables except for $x$ itself – are involved in a single expression.

In order to keep track of the arguments of functions but at the same time keep expressions readable, we add subscripts '$x$' or $i$ to an expression defined at the point $x$ or $x_i$, resp.

Moreover, we append similar subscripts to differential operators in order to indicate what point's coordinates they act on, and to inverse differential operators in order to indicate the second argument of the Green function involved. According to eqn. (1.15), the first argument of that Green function is the point at which the result of the action of the inverse operator is defined.

### 6.3.1 Variation

We first write eqn. (6.2) with the subscripts $x, 1, 2, 3$ introduced above:

$$S_{(3)} = \frac{1}{8\pi} \int \mathrm{d}^2x \sqrt{-g_x} \left[ \frac{\ln(\Box_2/\Box_3)}{\Box_2 - \Box_3} (\Box\phi)_2 (\Box\phi)_3 \right]_x \left[ \frac{1}{\Box_1} R_1 \right]_x$$

Now we carry out the variation, in general terms:

$$\begin{aligned} \delta S_{(3)} = \frac{1}{8\pi} \int \mathrm{d}^2x \Bigg\{ &(\delta\sqrt{-g_x}) \left[ \frac{\ln(\Box_2/\Box_3)}{\Box_2 - \Box_3} (\Box\phi)_2 (\Box\phi)_3 \right]_x \left[ \frac{1}{\Box_1} R_1 \right]_x \\ &+ \sqrt{-g_x} \left( \delta \frac{\ln(\Box_2/\Box_3)}{\Box_2 - \Box_3} (\Box\phi)_2 (\Box\phi)_3 \right)_x \left[ \frac{1}{\Box_1} R_1 \right]_x \\ &+ \sqrt{-g_x} \left[ \frac{\ln(\Box_2/\Box_3)}{\Box_2 - \Box_3} (\Box\phi)_2 (\Box\phi)_3 \right]_x \left( \delta \frac{1}{\Box_1} R_1 \right)_x \Bigg\} . \end{aligned} \tag{6.15}$$

Using the definition of $\frac{\ln(\Box_2/\Box_3)}{\Box_2 - \Box_3}$, eqn. (6.4), and the variation rule for $\sqrt{-g_x}$, eqn. (1.19), we obtain

$$\begin{aligned} \delta S_{(3)} = \frac{1}{8\pi} \int \mathrm{d}^2x \sqrt{-g_x} \Bigg\{ &\frac{1}{2} g_x^{\mu\nu} (\delta g_{\mu\nu\, x}) \left[ \frac{\ln(\Box_2/\Box_3)}{\Box_2 - \Box_3} (\Box\phi)_2 (\Box\phi)_3 \right]_x \left[ \frac{1}{\Box_1} R_1 \right]_x \\ &- \left( \delta \int_0^\infty \mathrm{d}m^2 \, \frac{1}{m^2 - \Box_2} \frac{1}{m^2 - \Box_3} (\Box\phi)_2 (\Box\phi)_3 \right)_x \left[ \frac{1}{\Box_1} R_1 \right]_x \\ &+ \left[ \frac{\ln(\Box_2/\Box_3)}{\Box_2 - \Box_3} (\Box\phi)_2 (\Box\phi)_3 \right]_x \left[ \left( \delta \frac{1}{\Box_1} \right) R_1 + \frac{1}{\Box_1} \delta R_1 \right]_x \Bigg\} . \end{aligned} \tag{6.16}$$

The symmetry of $\frac{\ln(\Box_2/\Box_3)}{\Box_2 - \Box_3}$ with respect to the points $2, 3$ allows us to write its variation as

$$\left(\delta\int_0^\infty \mathrm{d}m^2\,\frac{1}{m^2-\Box_2}\frac{1}{m^2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right)_x$$
$$= 2\int_0^\infty \mathrm{d}m^2\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_x\left(\delta\frac{1}{m^2-\Box_3}(\Box\phi)_3\right)_x$$
$$= 2\int_0^\infty \mathrm{d}m^2\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_x\left[\left(\delta\frac{1}{m^2-\Box_3}\right)(\Box\phi)_3+\frac{1}{m^2-\Box_3}(\delta\Box\phi)_3\right]_x \ .$$

We plug this expression into (6.16) and apply the variation rule for inverse wave operators, eqn. (1.22):

$$\delta S_{(3)} = \frac{1}{8\pi}\int \mathrm{d}^2x\,\sqrt{-g_x}\left\{\frac{1}{2}g_x^{\mu\nu}(\delta g_{\mu\nu\,x})\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_x\left[\frac{1}{\Box_1}R_1\right]_x\right.$$
$$-2\int_0^\infty \mathrm{d}m^2\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_x\left[\frac{1}{m^2-\Box_3}\left((\delta\Box_3)\left[\frac{1}{m^2-\Box_4}(\Box\phi)_4\right]_3+\delta\Box_3\phi_3\right)\right]_x\left[\frac{1}{\Box_1}R_1\right]_x$$
$$\left.+\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_x\left[\frac{1}{\Box_1}\left(-(\delta\Box_1)\left[\frac{1}{\Box_5}R_5\right]_1+\delta R_1\right)\right]_x\right\} \ .$$

Now we make use of the symmetry of Green functions, eqn. (1.17):

$$\delta S_{(3)} = \frac{1}{8\pi}\int \mathrm{d}^2x\,\sqrt{-g_x}\left\{\frac{1}{2}g_x^{\mu\nu}(\delta g_{\mu\nu\,x})\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_x\left[\frac{1}{\Box_1}R_1\right]_x\right.$$
$$-2\int_0^\infty \mathrm{d}m^2\left(\frac{1}{m^2-\Box_3}\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_3\left[\frac{1}{\Box_1}R_1\right]_3\right)_x(\delta\Box_x)\left(\frac{1}{m^2-\Box_4}(\Box\phi)_4+\phi\right)_x$$
$$\left.+\left(\frac{1}{\Box_1}\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_1\right)_x\left(-(\delta\Box_x)\left[\frac{1}{\Box_5}R_5\right]_x+\delta R_x\right)\right\} \ .$$

These steps have put us into a position where we vary only objects defined at the point $x$. Therefore, we can safely drop subscripts $x$ from now on.

In a last step, we apply the variation rules for $\delta\Box$, eqn. (1.21), and $\delta R$, eqn. (1.20); marked terms cancel due to the identity (2.2):

$$\delta S_{(3)} = \frac{1}{8\pi}\int \mathrm{d}^2x\,\sqrt{-g}\,(\delta g_{\mu\nu})\left\{\frac{1}{2}g^{\mu\nu}\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]\left[\frac{1}{\Box_1}R_1\right]\right.$$
$$-\int_0^\infty \mathrm{d}m^2\left[\!\!\left[2\left(\frac{1}{m^2-\Box_3}\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_3\left[\frac{1}{\Box_1}R_1\right]_3\right)^{(,\mu}\left(\frac{1}{m^2-\Box_4}(\Box\phi)_4+\phi\right)^{,\nu)}\right.\right.$$
$$-g^{\mu\nu}\left[\left(\frac{1}{m^2-\Box_3}\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_3\left[\frac{1}{\Box_1}R_1\right]_3\right)^{,\varepsilon}\left(\frac{1}{m^2-\Box_4}(\Box\phi)_4+\phi\right)_{,\varepsilon}\right.$$
$$\left.\left.+\left(\frac{1}{m^2-\Box_3}\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_3\left[\frac{1}{\Box_1}R_1\right]_3\right)\Box\left(\frac{1}{m^2-\Box_4}(\Box\phi)_4+\phi\right)\right]\right]\!\!\right]$$

$$
-\left(\frac{1}{\Box_1}\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_1\right)^{(;\mu}\left[\frac{1}{\Box_5}R_5\right]^{,\nu)}
+\frac{1}{2}g^{\mu\nu}\left[\left(\frac{1}{\Box_1}\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_1\right)^{,\varepsilon}\left[\frac{1}{\Box_5}R_5\right]_{,\varepsilon}
+\left(\frac{1}{\Box_1}\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_1\right)R\right]
+\left(\frac{1}{\Box_1}\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_1\right)^{;\mu\nu}-g^{\mu\nu}\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]
-R^{\mu\nu}\left(\frac{1}{\Box_1}\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_1\right)\Bigg\}. \quad (6.17)
$$

### 6.3.2 The stress tensor

Again we choose coordinates such that the diagonal metric elements vanish and consider the $(u,u)$ and, analogously, $(v,v)$ component of the energy momentum tensor. By definition (5.7), we extract $\langle T^{\mu\nu}\rangle$ from eqn. (6.17) and obtain

$$
\langle T^{uu}_{(3)}\rangle=\frac{1}{8\pi}\Bigg\{2\int_0^\infty \mathrm{d}m^2\,\left(\frac{1}{m^2-\Box_3}\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_3\left[\frac{1}{\Box_1}R_1\right]_3\right)_{,u}\left(\frac{1}{m^2-\Box_4}(\Box\phi)_4+\phi\right)_{,u}
+\left(\frac{1}{\Box_1}\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_1\right)_{,u}\left[\frac{1}{\Box_5}R_5\right]_{,u}-\left(\frac{1}{\Box_1}\left[\frac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}(\Box\phi)_2(\Box\phi)_3\right]_1\right)_{;uu}\Bigg\}.
$$

If we substitute the definition of $\dfrac{\ln(\Box_2/\Box_3)}{\Box_2-\Box_3}$, we can use the symmetry with respect to the points $2,3$ to write in shorthand notation

$$
\langle T^{uu}_{(3)}\rangle=\frac{1}{8\pi}\int_0^\infty \mathrm{d}m^2\,\Bigg\{2\left(\frac{1}{m^2-\Box_3}\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_3\left[\frac{1}{\Box_1}R_1\right]_3\right)_{,u}\left(\frac{1}{m^2-\Box_4}(\Box\phi)_4+\phi\right)_{,u}
+\left(\frac{1}{\Box_1}\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_1^2\right)_{,u}\left[\frac{1}{\Box_5}R_5\right]_{,u}-\left(\frac{1}{\Box_1}\left[\frac{1}{m^2-\Box_2}(\Box\phi)_2\right]_1^2\right)_{;uu}\Bigg\}.
$$

As in the case of the second-order correction, we can now express the inverse wave operators in terms of Green functions,

$$
\langle T^{uu}_{(3)}\rangle=\frac{1}{8\pi}\int_0^\infty \mathrm{d}m^2\,\Bigg\{2\Bigg[\int \mathrm{d}^2x_3\,\sqrt{-g(x_3)}\,G^m_{\mathrm{R}}(x,x_3)
\times\left(\int \mathrm{d}^2x_2\,\sqrt{-g(x_2)}\,G^m_{\mathrm{R}}(x_3,x_2)(\Box\phi)_2\right)\left(\int \mathrm{d}^2x_1\,\sqrt{-g(x_1)}\,G_{\mathrm{R}}(x_3,x_1)R_1\right)\Bigg]_{,u}
\times\left[\phi+\int \mathrm{d}^2x_4\,\sqrt{-g(x_4)},G^m_{\mathrm{R}}(x,x_4)(\Box\phi)_4\right]_{,u}
$$

$$
\begin{aligned}
&+\left[\int \mathrm{d}^2x_1\,\sqrt{-g(x_1)}\,G_{\mathrm{R}}(x,x_1)\left(\int \mathrm{d}^2x_2\,\sqrt{-g(x_2)}\,G^m_{\mathrm{R}}(x_1,x_2)(\Box\phi)_2\right)^2\right]_{,u}\\
&\qquad\qquad\qquad\qquad\times\left[\int \mathrm{d}^2x_5\sqrt{-g(x_5)}\,G_R(x,x_5)R_5\right]_{,u}\\
&-\left[\int \mathrm{d}^2x_1\,\sqrt{-g(x_1)}\,G_{\mathrm{R}}(x,x_1)\left(\int \mathrm{d}^2x_2\,\sqrt{-g(x_2)}\,G^m_{\mathrm{R}}(x_1,x_2)(\Box\phi)_2\right)^2\right]_{;uu}\Bigg\}\,. \qquad (6.18)
\end{aligned}
$$

Clearly, we are not better off with this expression than we were with (6.14); this one can also be solved only numerically. This is left for future work as well.

## 6.4 The tangential pressure

### 6.4.1 Variation with respect to the dilaton

While performing the variation with respect to the dilaton, we keep the two-dimensional metric fixed and independent from the dilaton. To the approximation made in [8] (neglecting fourth orders in curvatures), the second-order and third-order contributions to the effective action, (6.1) and (6.2), can be replaced by a term similar to (6.1) but with flat-spacetime differential operators [22],

$$
\bar{S} = -\frac{1}{8\pi}\int \mathrm{d}^2x\,(\bar{\Box}\phi)\ln\Big(\frac{-\bar{\Box}}{\mu^2}\Big)\phi\,. \qquad (6.19)
$$

In order to vary (6.19), we first integrate by parts and neglect the surface terms which are not varied. Then we interchange the gradient with the inverse d'Alembertian which is always possible in flat space.

$$
\begin{aligned}
\bar{S} &= \frac{1}{8\pi}\int \mathrm{d}^2x\,(\bar{\nabla}^\mu\phi)\bar{\nabla}_\mu\ln\Big(\frac{-\bar{\Box}}{\mu^2}\Big)\phi + (\text{surface terms})\\
&= \frac{1}{8\pi}\int \mathrm{d}^2x\,(\bar{\nabla}^\mu\phi)\ln\Big(\frac{-\bar{\Box}}{\mu^2}\Big)\bar{\nabla}_\mu\phi\,.
\end{aligned}
$$

By (1.17) and after raising and lowering indices, we write the variation as

$$
\begin{aligned}
\delta_\phi\bar{S} &= \frac{1}{4\pi}\int \mathrm{d}^2x\,(\delta_\phi\bar{\nabla}^\mu\phi)\ln\Big(\frac{-\bar{\Box}}{\mu^2}\Big)\bar{\nabla}_\mu\phi\\
&= \frac{1}{4\pi}\int \mathrm{d}^2x\,(\bar{\nabla}^\mu\delta_\phi\phi)\ln\Big(\frac{-\bar{\Box}}{\mu^2}\Big)\bar{\nabla}_\mu\phi\,.
\end{aligned}
$$

Now we again integrate by parts, neglect surface terms, and interchange the gradient with the inverse d'Alembertians to obtain

$$
\delta_\phi\bar{S} = \frac{-1}{4\pi}\int \mathrm{d}^2x\,(\delta_\phi\phi)\ln\Big(\frac{-\bar{\Box}}{\mu^2}\Big)\bar{\Box}\phi\,. \qquad (6.20)
$$

### 6.4.2 The tangential pressure

*Reading off from the variation of the effective action*

Let's first have a look at the expression $\bar{\Box}\phi$ in the Schwarzschild geometry. As $\phi = -\ln x^r$ in this case – see eqn. (5.13) and the following lines – we have

$$\bar{\Box}\phi = -\partial_\mu \eta^{\mu\nu} \partial_\nu \ln x^r = -\partial^2_{x^{r*}} \ln x^r = \left(1 - \frac{2M}{x^r}\right)\left(-\frac{1}{x^{r2}} + \frac{4M}{x^{r3}}\right) . \tag{6.21}$$

From eqn. (6.20), we can read off the $(\theta, \theta)$ component of the energy-momentum tensor or tangential pressure according to eqn. (5.9) and substitute eqn. (6.21),

$$\langle T^\theta_\theta(x)\rangle = \frac{-1}{32\pi^2 x^{r2}} \ln\left(\frac{-\bar{\Box}}{\mu^2}\right)\left(1 - \frac{2M}{y^r}\right)\left(-\frac{1}{y^{r2}} + \frac{4M}{y^{r3}}\right) .$$

*Performing the regularization*

Next, we want to explicitly perform the regularization we have implicitly assumed before by carrying along the parameter $\mu$. In order to do so, we start with the integral

$$\langle T^\theta_\theta(x)\rangle = \frac{1}{32\pi^2 x^{r2}} \int_0^\infty \mathrm{d}m^2 \, \frac{1}{m^2 - \Box}\left(1 - \frac{2M}{y^r}\right)\left(-\frac{1}{y^{r2}} + \frac{4M}{y^{r3}}\right) .$$

Here we have used the definition of $\ln \Box$ which corresponds to eqn. (6.3) up to the regularization parameter $\mu$ which, in (6.3), is only a symbolic notation. Next we write the expression just obtained in terms of Green functions and plug in the retarded Green function (6.10),

$$\begin{aligned}
\langle T^\theta_\theta(x)\rangle &= \frac{1}{32\pi^2 x^{r2}} \int_0^\infty \mathrm{d}m^2 \int \mathrm{d}^2 y \, G^m_{\mathrm{R}}(x,y)\left(1 - \frac{2M}{y^r}\right)\left(-\frac{1}{y^{r2}} + \frac{4M}{y^{r3}}\right) \\
&= \frac{1}{64\pi^2 x^{r2}} \int_0^\infty \mathrm{d}m^2 \int \mathrm{d}^2 y \, \Theta(x^t - y^t)\Theta(-\sigma_{xy}) J_0\left(m\sqrt{-2\sigma_{xy}}\right) \\
&\qquad \times \left(1 - \frac{2M}{y^r}\right)\left(-\frac{1}{y^{r2}} + \frac{4M}{y^{r3}}\right) .
\end{aligned} \tag{6.22}$$

This integral is divergent (see eqn. (6.24)); in order to regularize it, we introduce a small parameter $\varepsilon > 0$ and write

$$\begin{aligned}
\langle T^\theta_\theta(x)\rangle = \frac{1}{64\pi^2 x^{r2}} \lim_{\varepsilon\to 0} \int_0^\infty \mathrm{d}m^2 \int \mathrm{d}^2 y \, \Theta(x^t - y^t)\Theta(-\sigma_{xy}) J_0\left(m\sqrt{-2\sigma_{xy}}\right) J_0(m\varepsilon) \\
\times \left(1 - \frac{2M}{y^r}\right)\left(-\frac{1}{y^{r2}} + \frac{4M}{y^{r3}}\right) .
\end{aligned} \tag{6.23}$$

We can now make use of a formula which can be found, e.g., in [24]:

$$\int_0^\infty \mathrm{d}m^2 \, J_0\left(m\sqrt{-2\xi_1}\right) J_0\left(m\sqrt{-2\xi_2}\right) = 2\delta(\xi_1 - \xi_2) .$$

Plugging this into eqn. (6.23), that expression simplifies to

$$\langle T_\theta^\theta(x)\rangle = \frac{1}{32\pi^2 x^{r\,2}} \lim_{\varepsilon\to 0} \int \mathrm{d}^2 y\, \Theta(x^t - y^t)\Theta(-\sigma_{xy})\delta(\sigma_{xy}+\varepsilon/2) \left(1-\frac{2M}{y^r}\right)\left(-\frac{1}{y^{r\,2}}+\frac{4M}{y^{r\,3}}\right).$$

As $\varepsilon > 0$, the delta distribution selects only negative $\sigma_{xy}$ so there is no need for the $\Theta(-\sigma_{xy})$,

$$\begin{aligned}
\langle T_\theta^\theta(x)\rangle &= \frac{1}{32\pi^2 x^{r\,2}} \lim_{\varepsilon\to 0} \int \mathrm{d}^2 y\, \Theta(x^t - y^t)\delta\left(\frac{1}{2}\left[(y^{r^*}-x^{r^*})^2-(y^t-x^t)^2+\varepsilon\right]\right) \\
&\qquad\qquad \times \left(1-\frac{2M}{y^r}\right)\left(-\frac{1}{y^{r\,2}}+\frac{4M}{y^{r\,3}}\right) \\
&= \frac{1}{32\pi^2 x^{r\,2}} \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}y^{r^*} \int_{-\infty}^{\infty} \mathrm{d}y^t\, \Theta(x^t-y^t)\frac{-1}{y^t-x^t}\delta\left(y^t-x^t\pm\sqrt{(y^{r^*}-x^{r^*})^2+\varepsilon}\right) \\
&\qquad\qquad \times \left(1-\frac{2M}{y^r}\right)\left(-\frac{1}{y^{r\,2}}+\frac{4M}{y^{r\,3}}\right) \\
&= \frac{1}{32\pi^2 x^{r\,2}} \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}y^{r^*} \frac{1}{\sqrt{(y^{r^*}-x^{r^*})^2+\varepsilon}}\frac{1}{y^{r\,2}}\ . 
\end{aligned} \tag{6.24}$$

At this point we see clearly that the expression (6.22) (where $\varepsilon = 0$) is divergent at the point $y = x$.

Next, we must get rid of the parameter $\varepsilon$ by singling out and discarding the divergent terms. These terms are meant by the symbolic $\mu$ term in eqn. (6.3). We start with integrating by parts,

$$\begin{aligned}
\langle T_\theta^\theta(x)\rangle = \frac{1}{32\pi^2 x^{r\,2}} \lim_{\varepsilon\to 0}\Bigg\{ &\Bigg[\left(1-\frac{2M}{y^r}\right)\left(-\frac{1}{y^{r\,2}}+\frac{4M}{y^{r\,3}}\right) \\
&\qquad \times \ln\left|y^{r^*}-x^{r^*}+\sqrt{(y^{r^*}-x^{r^*})^2+\varepsilon^2}\right|\Bigg]_{y^{r^*}=-\infty}^{y^{r^*}=\infty} \\
+2\int_{-\infty}^{\infty} \mathrm{d}y^{r^*} \left(1-\frac{2M}{y^r}\right)&\left(\frac{2}{y^{r\,3}}-\frac{18M}{y^{r\,4}}+\frac{32M^2}{y^{r\,5}}\right)\ln\left|y^{r^*}-x^{r^*}+\sqrt{(y^{r^*}-x^{r^*})^2+\varepsilon^2}\right|\Bigg\}\ .
\end{aligned} \tag{6.25}$$

Both limits in the first term vanish as the divergence of the logarithm is suppressed by the coefficients each of which vanishes in one of the limits. This fact has also been checked using Maple.

We now integrate by parts again; for this, we first compute the integral of the logarithmic term,

$$\begin{aligned}
&\int \mathrm{d}y^{r^*}\, \ln\left|y^{r^*}-x^{r^*}+\sqrt{(y^{r^*}-x^{r^*})^2+\varepsilon^2}\right| \\
&\qquad = (y^{r^*}-x^{r^*})\ln\left|y^{r^*}-x^{r^*}+\sqrt{(y^{r^*}-x^{r^*})^2+\varepsilon^2}\right| - \sqrt{(y^{r^*}-x^{r^*})^2+\varepsilon^2}\ ,
\end{aligned}$$

then obtain from eqn. (6.25) where surface terms vanish:

$$\begin{aligned}
\langle T^{\theta}_{\theta}(x)\rangle = \frac{1}{32\pi^2 x^{r2}} \lim_{\varepsilon\to 0} \Bigg\{ \Bigg[ \Bigg( & (y^{r^*} - x^{r^*}) \ln\left| y^{r^*} - x^{r^*} + \sqrt{(y^{r^*} - x^{r^*})^2 + \varepsilon^2} \right| \\
& - \sqrt{(y^{r^*} - x^{r^*})^2 + \varepsilon^2} \Bigg) \left(1 - \frac{2M}{y^r}\right) \left(\frac{2}{y^{r3}} - \frac{18M}{y^{r4}} + \frac{32M^2}{y^{r5}}\right) \Bigg]_{y^{r^*}=-\infty}^{y^{r^*}=\infty} \\
- \int_{-\infty}^{\infty} \mathrm{d}y^{r^*} & \left( (y^{r^*} - x^{r^*}) \ln\left| y^{r^*} - x^{r^*} + \sqrt{(y^{r^*} - x^{r^*})^2 + \varepsilon^2} \right| - \sqrt{(y^{r^*} - x^{r^*})^2 + \varepsilon^2} \right) \\
& \times \left(1 - \frac{2M}{y^r}\right) \left( -\frac{3}{y^{r4}} + \frac{44M}{y^{r5}} - \frac{170M^2}{y^{r6}} + \frac{192M^3}{y^{r7}} \right) \Bigg\} \qquad (6.26)
\end{aligned}$$

Again, the surface terms vanish because the logarithmic divergence is too weak.

Finally, we deal with the parameter $\varepsilon$ by expanding (6.26) in powers of $\varepsilon$ and sending $\varepsilon$ to zero afterwards.

The expansion of the logarithm in powers of $\varepsilon$ in the vicinity of $\varepsilon = 0$ reads

$$\begin{aligned}
& \ln\left| y^{r^*} - x^{r^*} + \sqrt{(y^{r^*} - x^{r^*})^2 + \varepsilon^2} \right| \\
& = \begin{cases} \ln\left(2|y^{r^*} - x^{r^*}|\right) + \dfrac{1}{4}\dfrac{\varepsilon^2}{(y^{r^*} - x^{r^*})^2} + \mathscr{O}(\varepsilon^4) & \text{if } y^{r^*} > x^{r^*} \\ \left[-\ln\left(2|y^{r^*} - x^{r^*}|\right) + 2\ln\varepsilon\right] - \dfrac{1}{4}\dfrac{\varepsilon^2}{(y^{r^*} - x^{r^*})^2} + \mathscr{O}(\varepsilon^4) & \text{if } y^{r^*} < x^{r^*} \end{cases} . \qquad (6.27)
\end{aligned}$$

Furthermore, introduce shorthand notation for polynomial term,

$$V(y^r) = -\frac{3}{y^{r4}} + \frac{44M}{y^{r5}} - \frac{170M^2}{y^{r6}} + \frac{192M^3}{y^{r7}} \; .$$

Inserting eqn. (6.27) in eqn. (6.26) with the surface terms gone, the only $\varepsilon$ independent terms are

$$\begin{aligned}
\langle T^{\theta}_{\theta}(x)\rangle = -\frac{1}{32\pi^2 x^{r2}} \int_{-\infty}^{\infty} \mathrm{d}y^{r^*} & \left( |y^{r^*} - x^{r^*}| \ln\left(2|y^{r^*} - x^{r^*}|\right) - |y^{r^*} - x^{r^*}| \right) \\
& \times \left(1 - \frac{2M}{y^r}\right) V(y^r) \; . \qquad (6.28)
\end{aligned}$$

*Evaluation of the tangential pressure*

Using the definition of the tortoise coordinate, eqn. (2.5), in eqn. (6.28), we obtain

$$\begin{aligned}
\langle T^{\theta}_{\theta}(x)\rangle &= -\frac{1}{32\pi^2 x^{r2}} \int_{2M}^{\infty} \mathrm{d}y^r \, |y^{r^*} - x^{r^*}| \ln\left(\frac{2}{e}|y^{r^*} - x^{r^*}|\right) V(y^r) \\
&= -\frac{1}{32\pi^2 x^{r2}} \int_{2M}^{\infty} \mathrm{d}y^r \, |y^{r^*} - x^{r^*}| \ln\left(\frac{2}{e}|x^{r^*}| \left|1 - \frac{y^{r^*}}{x^{r^*}}\right|\right) V(y^r)
\end{aligned}$$

$$= -\frac{1}{32\pi^2 x^{r2}} \left\{ \ln\left(\frac{2}{e}|x^{r^*}|\right) \int\limits_{2M}^{\infty} \mathrm{d}y^r \, |y^{r^*} - x^{r^*}| V(y^r) \right.$$

$$\left. + \int\limits_{2M}^{\infty} \mathrm{d}y^r \, |y^{r^*} - x^{r^*}| V(y^r) \ln\left|1 - \frac{y^{r^*}}{x^{r^*}}\right| \right\} . \tag{6.29}$$

Let's call the first part of this expression $I_1$,

$$I_1(x) = -\frac{1}{32\pi^2 x^{r2}} \ln\left(\frac{2}{e}|x^{r^*}|\right) \int\limits_{2M}^{\infty} \mathrm{d}y^r \, |y^{r^*} - x^{r^*}| V(y^r) \; . \tag{6.30}$$

It can be exactly evaluated. First, we note that

$$\int\limits_{2M}^{\infty} \mathrm{d}y^r \, V(y^r) = 0 \qquad \text{and} \qquad \int\limits_{2M}^{\infty} \mathrm{d}y^r \, V(y^r) y^{r^*} = 0 \; .$$

Combining these yields

$$\int\limits_{2M}^{\infty} \mathrm{d}y^r \, (y^{r^*} - x^{r^*}) V(y^r) = 0 \; .$$

Now we split the integration interval at $y^r = x^r$ to obtain

$$\int\limits_{2M}^{x^r} \mathrm{d}y^r \, (y^{r^*} - x^{r^*}) V(y^r) = -\int\limits_{x^r}^{\infty} \mathrm{d}y^r \, (y^{r^*} - x^{r^*}) V(y^r) = \int\limits_{x^r}^{\infty} \mathrm{d}y^r \, (x^{r^*} - y^{r^*}) V(y^r)$$

$$\text{and thus} \qquad \int\limits_{2M}^{\infty} \mathrm{d}y^r \, |y^{r^*} - x^{r^*}| V(y^r) = 2 \int\limits_{x^r}^{\infty} \mathrm{d}y^r \, (y^{r^*} - x^{r^*}) V(y^r) \; .$$

Using this relation in $I_1$ as given by eqn. (6.30) yields

$$\begin{aligned} I_1(x^r) &= -\frac{1}{32\pi^2 x^{r2}} \ln\left(\frac{2}{e}|x^{r^*}|\right) \int\limits_{2M}^{\infty} \mathrm{d}y^r \, |y^{r^*} - x^{r^*}| V(y^r) \\ &= -\frac{1}{16\pi^2 x^{r2}} \ln\left(\frac{2}{e}|x^{r^*}|\right) \int\limits_{x^r}^{\infty} \mathrm{d}y^r \, (y^{r^*} - x^{r^*}) V(y^r) \\ &= \frac{1}{32\pi^2} \ln\left(\frac{2}{e}|x^{r^*}|\right) \left(\frac{M^2}{x^{r4}} - 6\frac{M^3}{x^{r5}} + 8\frac{M^4}{x^{r6}}\right) \; . \end{aligned} \tag{6.31}$$

The last integral has been solved with the help of MAPLE.

Let's note that expression (6.31) vanishes in the vicinity of the horizon as well as far away from the Black Hole.

The second remainder of expression (6.29) can not be handled as easily, and probably it can be evaluated only numerically. As before, the numerical calculations are not part of this diploma thesis but are left for future investigations.

In order to decide on the form of the complete $\langle T_\theta^\theta(x)\rangle$, at least numerical results for that second part must be available. In particular, it would be important to know whether the logarithm involved gives rise to a divergence as $x$ approaches the event horizon.

Chapter 7

# Conclusion and Outlook

The subject of this diploma thesis has been the radiation of Black Holes discovered by S. Hawking and the definition of two-dimensional effective action models supposed to allow for an insight into problems not solvable in a four-dimensional theory, e.g. the problem of back-reaction of the radiation on the geometry of the Black Hole spacetime. The question at hand is how such a model must be constructed in order to qualitatively yield the same Hawking radiation as obtained in real four-dimensional spacetime.

In the first part of this work, we have given a review of the spacetime of Black Holes and explained how particles can be created by a non-Minkowskian spacetime. Then we have had a look at several explanations of Hawking radiation, including an intuitive picture as well as an outline of Hawking's own derivation.

In the second part, we have introduced two-dimensional models of spherically symmetric Black Hole radiation and discussed some models where the effective action is determined completely by the conformal anomaly of the quantum field. In particular, it was one of the tasks of this work to investigate an apparent discrepancy between several of these effective action models. I convinced myself that the solution was contained in a publication by J.S. Dowker [5] which turned out to be widely accepted by that time.

These models all assume a gravitational action obtained by dimensional reduction from the one for four dimensions. However, they are unsatisfactory because of the matter part. The simpler model yields a Hawking flux which qualitatively agrees with the four-dimensional one, but it is questionable because the matter part has no four-dimensional origin. The dilaton model does have a four-dimensional origin of the matter but yields qualitatively unsatisfactory results for the Hawking radiation.

There are approaches to cure this disagreement by adding conformally invariant terms to the effective action. The proposal given recently by Y. Gusev and A. Zelnikov [8] is investigated in detail. We were able to calculate its contributions to the matter-related energy-momentum tensor up to the point where a numerical treatment is necessary. This necessity is mostly due to the fact that terms depending on the Schwarzschild radial coordinate $r$ have to be integrated with respect to the coordinate $r^*$ (see eqn. (2.5)) which is a function of $r$ the inverse of which – known as the Lambert W function – is available only tabularized.

These results are equations (6.14), (6.18), and (6.29). A part of the component given in the latter can be exactly calculated, see eqn. (6.31). However, the other part would be needed as well in order to be able to decide on the nature, e.g. the asymptotic behaviour, of the whole expression.

The components of the energy-momentum tensor have a non-local nature. Thus it is possible that they contribute to the Hawking flux which, due to the non-locality of the quantum state, is a non-local effect as well.

Once numerical results are available, it will be possible to decide what contributions the correction proposed by Gusev and Zelnikov makes to the effective action model and whether they are able to restore qualitative agreement with the four-dimensional model as for the Hawking flux.

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