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Low-dimensional Hot and Dense Fermionic Systems in an External Magnetic Field

Marcel Horstmann

Friedrich-Schiller-Universität Jena
Physikalisch-Astronomische Fakultät

> *"In den Werken der Natur und gerade in ihnen herrscht die Regel, nicht blinder Zufall. Der Endzweck aber, um dessentwillen ein Ding geschaffen oder geworden ist, liegt im Bereich des Schönen."*
>
> – Aristoteles

Erstgutachter:	Prof. Dr. Andreas Wipf
Zweitgutachter:	M. Sc. Michael Mandl

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Abstract

The Gross-Neveu model in $1+1$ dimensions in the large N_f limit breaks chiral symmetry spontaneously, leading to a dynamically generated fermion mass. For non-vanishing chemical potential and temperature, a non-trivial phase structure emerges with regions of intact and spontaneously broken symmetry respectively. Depending on the external parameters the transition between these regions can be first or second order. In a constant external magnetic field we find the phenomenon of magnetic catalysis, which breaks the chiral symmetry even at the weakest interaction between the fermions. Further, we find the enhancement of the condensate for large B.

1 Introduction

In 1974, David J. Gross and André Neveu introduced a Lagrangian model to describe the theory of strong interactions [1]. Today the so-called Gross-Neveu model acts as a toy model for quantum chromodynamics (QCD). Further, it was found that this model has some applications in solid state physics, e. g. it describes first-order phase transitions of trans-polyacetylen with a coupled chemical potential ($(\mathrm{CH})_x$) [2]. Furthermore, the model can be used to describe some properties of quantum quasiparticles and particles in condensed matter physics or nuclear physics via the magnetic catalysis effect. Magnetic catalysis is the enhencement of dynamical chiral or flavor symmetry breaking by an external magnetic field strength coupled to the model [3].

In this thesis we investigate the partition function of the Gross-Neveu Lagrangian in the large flavor number limit ($N_f \to \infty$). We first investigate the symmetries of this model and find that chiral symmetry only is intact for certain external conditions.

The aim of the study is to derive different effective potential densities that are highly dependent on external parameters we couple to the model: a temperature T, a chemical potential μ and an external magnetic field B. Chiral symmetry is spontaneously broken if the chiral condensate $\left\langle \bar{\psi}\psi \right\rangle$ is non-vanishing. For different scenarios (e. g. including different external parameters) our main goal is to derive this effective potential by using the ζ-function renormalization prescription. After mapping out the phase diagram for the external parameters, we can make statements about phase transitions and their orders when said parameters are varied.

2 Theoretical Foundations

We introduce the basic formalism we use in order to formulate and discuss fermion interactions. First we introduce *four fermion theories* (4FT) and their symmetries. We are especially interested in chiral symmetry (and its breaking), because this will enable us to map out phase diagrams.

In the entirety of this work we use the conventions listed below.
All equations are written in natural units,

$$c = \hbar = k_B = 1\,. \tag{2.1}$$

Further, we only discuss problems in 1+1-dimensional spacetime. The x_0-component stands for the time and the x_1-component for the space dimension. In order to discuss thermodynamics we work in EUCLIDEAN spacetime exclusively. The spacetime is here isomorphic to the EUCLIDEAN space $\mathbb{R}^2$ with the metric tensor

$$\eta = \mathrm{diag}(+,+) \quad \Longleftrightarrow \quad \eta^{\mu\nu} = \delta^{\mu\nu}. \tag{2.2}$$

Whenever we are interested in finite-temperature phenomena, we work with the MATSUBARA formalism. The temporal direction is restricted to a finite interval $[0, \beta]$ where β can be interpreted as the inverse temperature, $\beta = \frac{1}{T}$. We will assume for the spatial direction to to have infinite extent in general. However, we will introduce the spatial extent L explicitly at some points during our calculations, which we then let tend to infinity implicitly.
We use the FEYNMANN slash notation

$$\not{A} := \gamma^\mu A_\mu \quad (= \gamma^0 A_0 + \gamma^1 A_1)\,, \tag{2.3}$$

where γ are the gamma matrices in 2 dimensions.
The gamma matrices are defined by the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}\mathbb{1}_2. \tag{2.4}$$

The bar over the DIRAC spinors defines its DIRAC adjoint

$$\bar{\psi} := \psi^\dagger\gamma^0\,. \tag{2.5}$$

2.1 Four Fermion Theories

Theories that describe the physical properties of four interacting Dirac fermions are called *four fermion theories* (4FT).
Gross and Neveu introduced a model in two dimensions with analogies to the theory of strong interactions. The Gross-Neveu model describes four fermion interactions in 1 + 1-dimensions with the Lagranian [1]:

$$\begin{aligned}\mathscr{L}_{\text{Gross}-\text{Neveu}} &= \mathscr{L}_{\text{Kinetic}} + \mathscr{L}_{\text{Interaction}} \\ &= \sum_{a=1}^{N_f} \overline{\psi_a}(\not{\partial} + m)\psi_a - \frac{g^2}{2N_f}\sum_{a=1}^{N_f}(\overline{\psi_a}\psi_a)^2 \\ &=: \bar{\psi}(\not{\partial} + m)\psi - \frac{g^2}{2N_f}(\bar{\psi}\psi)^2.\end{aligned} \tag{2.6}$$

N_f denotes the flavour number, m the mass and g^2 is a dimensionless coupling constant for strong interactions. The Dirac spinors ψ and $\overline{\psi}$ hold all the information about the fermions.

2.1.1 Symmetries of four fermion theories

One of the best ways to understand a physical or mathematical system is to study its symmetries in depth. Since the main focus of this thesis is the study of the Gross-Neveu model, we should first list its known symmetries. First, we mention the external symmetries:

1. Poincaré symmetry: This symmetry is the group of all general Lorentz transformations

   $$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu, \tag{2.7}$$

   with $\Lambda \in SO(2)$, in combination with spacetime translations

   $$x^\mu \to x'^\mu = x^\nu + c^\mu\,. \tag{2.8}$$

   Hence the Gross-Neveu model is invariant under special relativity transformations.

2. Parity and time-reversal symmetry: Since the Dirac spinor fields only occur in quadratic terms, the sign of any spatial or time coordinate can just be flipped

   $$(x_0, x_1) \to (x_0, -x_1) \qquad \text{(parity)} \tag{2.9}$$

   $$(x_0, x_1) \to (-x_0, x_1) \qquad \text{(time-reversal)}\,, \tag{2.10}$$

   both of which are $\mathbb{Z}_2$ symmetries.

3. Charge conjugation: Another $\mathbb{Z}_2$ symmetry that the Gross-Neveu model obeys is the charge conjugation of the spinors.

Aside from external symmetries, the Gross-Neveu model also holds some internal symmetries:

1. Flavor symmetry: This model was initially designed to be a toy-model for the strong interactions in quantum chromodynamics. To fulfill this symmetry the model must be invariant under flavor rotations

   $$\psi^a \to U^{ab}\psi^b \quad \text{and} \quad \bar{\psi}^a \to \bar{\psi}^b \left(U^\dagger\right)^{ba}, \tag{2.11}$$

   such that $U^\dagger U = \mathbb{1}_{N_f}$.

2. Chiral symmetry: In contrast to other four-fermio models, the (massless) Gross-Neveu model is only invariant under a discrete chiral symmetry. Like some of the external symmetries, this discrete chiral symmetry is isomorphic to $\mathbb{Z}_2$:

   $$\psi \to \gamma_5\psi \qquad\qquad \bar{\psi} \to -\bar{\psi}\gamma_5. \tag{2.12}$$

   For non-vanishing mass, this symmetry is explicitly broken.

The study of symmetries also shows that a continuous symmetry of a model always implies the existence of a conserved quantity. This concept is described by the Noether theorem, according to which every continuous symmetry transformation of the action implies a conserved Noether current and for vanishing spatial boundaries also a Noether charge.

2.1.2 Broken chiral symmetry

Via the Hubbard-Stratonovich transformation we convert the Gross-Neveu theory described by eq. (2.6) into its respective field theory by introducing an auxiliary field σ:

$$\mathscr{L} = \bar{\psi}\left(\sigma + \partial\!\!\!/\right)\psi - \frac{N_f}{2g^2}\sigma^2. \tag{2.13}$$

We can perform the discrete chiral transformation (2.12) and find that for a non-vanishing vacuum expectation value of the auxiliary field σ a mass term for the spinor fields ψ appears. This mass term, being proportional to the chiral condensate $\left\langle\bar{\psi}\psi\right\rangle$, then signal the spontaneous symmetry breaking.

2.2 Riemann ζ-function regularization

The main goal of this thesis is to calculate effective potentials for different external parameters. In order to do this we need renormalization and regularization techniques to deal with infinities which arise in our calculations.
Since we deal with operator determinants when calculating the effective potential, we choose the convenient method of ζ-function regularization [4]. First we consider the eigenvalue equation for an arbitrary DIRAC operator D

$$D\psi_n = \lambda\psi_n \,. \tag{2.14}$$

The associated ζ-function to this DIRAC operator is defined by:

$$\zeta(s) := \mathrm{Tr}\left(\frac{1}{D^s}\right) = \sum_n \frac{1}{\lambda_n^s}. \tag{2.15}$$

By taking the derivative with respect to s,

$$\frac{\partial}{\partial s}\zeta(s) = \sum_n \frac{\partial}{\partial s}\exp\left(-\ln\left(\lambda_n\right)s\right) = -\sum_n \frac{\ln\left(\lambda_n\right)}{\lambda_n^s}, \tag{2.16}$$

and considering $s = 0$, we obtain the relation

$$-\zeta'(0) = \sum_n \ln\left(\lambda_n\right) = \ln\left(\prod_n \lambda_n\right) = \ln\left(\det D\right), \tag{2.17}$$

such that the determinant of the operator D can also be defined as

$$\det D := \exp\left(-\zeta'(0)\right). \tag{2.18}$$

2.2.1 Heat kernel

Next we introduce the heat kernel of an operator D, which is linked to the associated ζ function. The heat kernel trace of D is defined via

$$\begin{aligned} K(t) &:= \mathrm{Tr}\left(\exp\left(-Dt\right)\right) && (2.19)\\ &= \sum_n \exp\left(-\lambda_n t\right). && (2.20) \end{aligned}$$

To link eq. (2.20) with the associated ζ-function we first use a MELLIN integral transformation defined by

$$\left\{MK\right\}(s) = \int_0^\infty \mathrm{d}t\, t^{s-1}K(t)\,. \tag{2.21}$$

This leads to the following relation between the heat kernel trace and the associated ζ-function of D:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} K(t). \tag{2.22}$$

2.2.2 Analytic continuation

Since the derivative of the ζ-function in eq. (2.22) does not necessarily converge for $s = 0$, it is challenging to calculate this value. In order to bypass this convergence problem we use the complex analysis theorem of analytical continuation, which reads:

Analytical continuation: Let U be a connected open subset (domain) of the complex numbers and $\tilde{f} : U \to \mathbb{C}$ a holomorphic function on U. Then there exists a unique function $f : \mathbb{C} \to \mathbb{C}$ which is holomorphic on the whole complex plane and matches with $\tilde{f}$ on U ($f|_U = \tilde{f}$). The function f is called analytic continuation of $\tilde{f}$.

We can make use of this theorem in the computation of the integral in eq. (2.22), because the heat kernel and the t^{s-1}-part are both analytic functions.

3 Mapping out the phase diagrams

The Gross-Neveu model (in Euclidean spacetime and after introducing the auxiliary field σ via a Hubbard-Stratonovich transformation) is described by the Lagrangian

$$\mathscr{L} = \bar{\psi}\left(\sigma + \not{\partial} + \mathrm{i}e\not{A} - \gamma_0\mu\right)\psi + \frac{N_f}{2g^2}\sigma^2, \tag{3.1}$$

where $e > 0$ is the elementary electric charge, A_μ is the vector potential describing an external magnetic field, μ denotes the chemical potential, N_f is the number of fermion flavors, and g^2 is the coupling constant of the 4-fermion interaction. We have introduced μ and A_μ in the standard way. Note that in eq. (3.1) summation over Dirac and flavor indices is implied. We set the quark mass $m = 0$ to preserve chiral symmetry. The partition function of this theory is given by the path integral

$$Z = \int D\left[\bar{\psi}, \psi\right] D[\sigma] \exp\left(-S\left[\bar{\psi}, \psi, \sigma\right]\right), \tag{3.2}$$

where the action in $d = 1 + 1$ dimensions is given by

$$S = \int \mathrm{d}^2x\, \mathscr{L}. \tag{3.3}$$

For the case of a homogeneous field $\sigma(x) = \sigma$ we obtain the partition function

$$Z = \int D\left[\bar{\psi}, \psi\right] D[\sigma] \exp\left(-\int \mathrm{d}^2x\, \bar{\psi} D\psi - \frac{N_f V_{\mathrm{st}}}{2g^2}\sigma^2\right), \tag{3.4}$$

where $V_{\mathrm{st}} = \int \mathrm{d}^2x$ denotes the spacetime volume and we have introduced the Dirac operator

$$D = \sigma + \not{\partial} + \mathrm{i}e\not{A} - \gamma_0\mu. \tag{3.5}$$

For a constant homogeneous magnetic field in $1 + 1$ dimensions the field strength tensor reads

$$F_{01} := \partial_0 A_1 - \partial_1 A_0 \overset{!}{=} B = -F_{10}, \tag{3.6}$$

with F_{00} and F_{11} vanishing. We choose the following gauge:

$$A_0 = -Bx_1 \text{ and } A_1 = 0\,. \tag{3.7}$$

The Dirac operator now has the following form:

$$D = \sigma + \gamma_0(\partial_0 - \mathrm{i}eBx_1 - \mu) + \gamma_1\partial_1. \tag{3.8}$$

Since only terms that are bilinear in ψ and $\bar{\psi}$ are left, we can integrate out the fermion fields. Every fermionic flavor can be integrated over separatly (using Gaussian integration for Grassmann variables) and each integration gives the determinant of any diagonal operator P:

$$\begin{aligned}\int D\left[\bar{\psi},\psi\right]\exp\left(-\bar{\psi}P\psi\right) &= \int\left(\prod_{i=1}^{N_f} D\left[\bar{\psi}_i,\psi_i\right]\right)\exp\left(\sum_{i,j=1}^{N_f}\bar{\psi}_i P_{ij}\psi_j\right)\\ &= \prod_{i=1}^{N_f}\int D\left[\bar{\psi}_i,\psi_i\right]\exp\left(-\bar{\psi}_i\tilde{P}\psi_i\right)\\ &= \prod_{i=1}^{N_f}\det\left(\tilde{P}\right) = \left(\det\left(\tilde{P}\right)\right)^{N_f}.\end{aligned} \tag{3.9}$$

where we have abused that P is diagonal in flavor space, $P_{ij} = \tilde{P}\delta_{ij}$. Using eq. (3.9) we can rewrite eq. (3.4) such that

$$Z = \int D[\sigma]\exp\left(-\frac{N_f V_{\mathrm{st}}}{2g^2}\sigma^2 + N_f\ln\left(\det(D)\right)\right) \tag{3.10}$$

$$= \int D[\sigma]\exp\left(-N_f V_{\mathrm{eff}}(\sigma)\right). \tag{3.11}$$

In the partition function above we have introduced the effective potential for σ, i. e.

$$V_{\mathrm{eff}}(\sigma) = \frac{V_{\mathrm{st}}}{2g^2}\sigma^2 - \ln\left(\det(D)\right). \tag{3.12}$$

The transformation $\sigma \to -\sigma$ of the auxiliary field leaves the effective potential invariant ($\mathbb{Z}_2$-symmetry), corresponding to the chiral symmetry of the Lagrangian.

We now consider the limit of an infinite number of flavors ($N_f \to \infty$). Only the minimum of $V_{\mathrm{eff}}(\sigma)$ contributes to the partition function (3.11) in this case. Hence, our main interest is to find the auxiliary field $\sigma = \sigma_0$ at which said minimum occurs. The chiral symmetry of (3.1) is only conserved if $\sigma_0 = 0$, however it is spontaneously broken if $\sigma_0 \neq 0$, in which case degenerate ground states exist.
Therefore, the main task will be to investigate the effective potential V_{eff} for different temperatures, chemical potentials and magnetic fields, and to find the corresponding minima in σ in order to map out phase diagrams in T-μ-B-space depicting regions with intact and sponteneously broken symmetry respectively.

3.1 Effective potential for $B = 0, \mu = 0, T = 0$

We start our computation of the effective potential with the simplest case, in which all external parameters are set to zero. The DIRAC operator of eq. (3.8) now reduces to the following simple form

$$D = \sigma + \not{\partial}. \tag{3.13}$$

In order to calculate the effective potential (3.12) we are interested in calculating $\ln(\det(D))$. We start off by rewriting

$$\det(D) = \sqrt{\det(D)^2}, \tag{3.14}$$

and abusing the fact that $\gamma_5 D \gamma_5 = \sigma - \not{\partial}$. Since $\det(\gamma_5) = 1$, we can rewrite

$$\det(D) = \sqrt{\det(D)\det(\gamma_5)\det(D)\det(\gamma_5)} = \sqrt{\det(D\gamma_5 D\gamma_5)}\,, \tag{3.15}$$

such that

$$\ln(\det(D)) = \frac{1}{2}\ln\left(\det\left(D\gamma_5 D\gamma_5\right)\right) = \frac{1}{2}\ln\left(\det\left(\sigma^2 - \partial^2\right)\right). \tag{3.16}$$

We use the zeta-function renormalization formalism to calculate $\det\left(\sigma^2 - \partial^2\right)$. To this end we need the eigenvalues λ_p of $\sigma^2 - \partial^2$, which are given by

$$\lambda_p = \sigma^2 + p_0^2 + p_1^2, \tag{3.17}$$

where p_μ denotes the momentum in the μ direction. Note that the eigenvalues are doubly degenerate.

Heat kernel: For the zeta-function renormalization we first calculate the heat kernel $K(t)$ of the the second order operator $D\gamma_5 D\gamma_5$ by using the eigenvalues λ_p:

$$\begin{aligned} K(t) &= \mathrm{Tr}\left\{\exp\left(-D\gamma_5 D\gamma_5 t\right)\right\} && (3.18)\\ &= \frac{2V_{\mathrm{st}}}{(2\pi)^2}\int \mathrm{d}^2p \exp\left(-\lambda_p t\right) && (3.19)\\ &= \frac{V_{\mathrm{st}}}{2\pi^2}\exp\left(-\sigma^2 t\right)\int \mathrm{d}p_0 \exp\left(-p_0^2 t\right)\int \mathrm{d}p_1 \exp\left(-p_1^2 t\right) && (3.20)\\ &= \frac{V_{\mathrm{st}}}{2\pi^2}\exp\left(-\sigma^2 t\right)\frac{\pi}{t} && (3.21)\\ &= \frac{V_{\mathrm{st}}}{2\pi}\frac{\exp\left(-\sigma^2 t\right)}{t}. && (3.22)\end{aligned}$$

ζ-function renormalization: Using a Mellin transformation we can find the ζ-function

$$\begin{aligned}\zeta(s) &= \frac{1}{\Gamma(s)}\int\limits_0^\infty \mathrm{d}t\, t^{s-1}K(t) && (3.23)\\ &= \frac{V_{\mathrm{st}}}{2\pi\Gamma(s)}\int\limits_0^\infty \mathrm{d}t\, t^{s-2}\exp\left(-\sigma^2 t\right) && (3.24)\\ &= \frac{V_{\mathrm{st}}}{2\pi}\sigma^{2-2s}\frac{\Gamma(s-1)}{\Gamma(s)} && (3.25)\\ &= \frac{V_{\mathrm{st}}}{2\pi}\frac{\sigma^{2-2s}}{s-1}. && (3.26)\end{aligned}$$

We solved the integral in eq. (3.24) using integral tables [5]. In eq. (3.24) we used the complex analysis concept of analytic continuation, allowing us to extend the part of the result that is holomorphic in an open region to the whole complex plane $\mathbb{C}$.

The determinant $D\gamma_5 D\gamma_5$ is linked to the ζ-function via the following relation,

$$\det\left(D\gamma_5 D\gamma_5\right) := \exp\left(-\zeta'(0)\right). \tag{3.27}$$

The derivative of ζ is

$$\zeta'(s) = -\frac{V_{\mathrm{st}}}{2\pi}\frac{\sigma^{2-2s}}{s-1}\left(\ln\left(\sigma^2\right) + \frac{1}{s-1}\right), \tag{3.28}$$

and its value at $s = 0$ is

$$\zeta'(0) = \frac{V_{\mathrm{st}}}{2\pi}\sigma^2(\ln\left(\sigma^2\right) - 1). \tag{3.29}$$

Note that we are taking the logarithm of a dimensionful quantity here. This can be remedied by adding a suitable counterterm containing the logarithm $\ln\left(\mu_0^2\right)$ of some dimensionful scale μ_0. For now we assume implicitly that this has been done and omit including μ_0 explicitly. We will come back to this point later.

Effective potential: We have finally found the effective potential (3.12),

$$\begin{aligned}V_{\mathrm{eff}}(\sigma) &= \frac{V_{\mathrm{st}}}{2g^2}\sigma^2 - \frac{1}{2}\ln\left(\exp\left(-\zeta'(0)\right)\right) && (3.30)\\ &= \frac{V_{\mathrm{st}}}{2g^2}\sigma^2 + \frac{V_{\mathrm{st}}}{4\pi}\sigma^2(\ln\left(\sigma^2\right) - 1) && (3.31)\\ &= \frac{V_{\mathrm{st}}}{2}\sigma^2\left(\frac{1}{g^2} - \frac{1}{2\pi} + \frac{\ln(\sigma^2)}{2\pi}\right), && (3.32)\end{aligned}$$

such that

$$\frac{V_{\mathrm{eff}}}{V_{\mathrm{st}}} = \frac{\sigma^2}{2}\left(\frac{1}{g^2} - \frac{1}{2\pi} + \frac{\ln(\sigma^2)}{2\pi}\right). \tag{3.33}$$

Let us now reintroduce the scale μ_0:

$$\frac{V_{\mathrm{eff}}}{V_{\mathrm{st}}} = \frac{\sigma^2}{2}\left(\frac{1}{g^2} - \frac{1}{2\pi} + \frac{1}{2\pi}\ln\left(\frac{\sigma^2}{\mu_0^2}\right)\right) \tag{3.34}$$

To determine the minimum point σ_0 of the effective potential density $V_{\mathrm{eff}}(\sigma)/V_{\mathrm{st}}$, we require:

$$\frac{\partial}{\partial\sigma}\frac{V_{\mathrm{eff}}(\sigma)}{V_{\mathrm{st}}}\bigg|_{\sigma=\sigma_0} = \sigma\left(\frac{1}{g^2} + \frac{1}{2\pi}\ln\left(\frac{\sigma^2}{\mu_0^2}\right)\right)\bigg|_{\sigma=\sigma_0} \overset{!}{=} 0\,, \tag{3.35}$$

leading to

$$\sigma_0 = 0 \quad \text{or} \quad \frac{1}{g^2} + \frac{1}{2\pi}\ln\left(\frac{\sigma_0^2}{\mu_0^2}\right) = 0\,. \tag{3.36}$$

We are only interested in the global minimum of the gap equation (3.35). It turns out that it is given by the non-trivial minimum positions

$$\sigma_0 = \mu_0\,\exp\left(-\frac{\pi}{g^2}\right) =: m_0 \tag{3.37}$$

$$\sigma_0' = -\mu_0\,\exp\left(-\frac{\pi}{g^2}\right)\,. \tag{3.38}$$

To verify that $V_{\mathrm{eff}}(\sigma_0)$ is indeed a minimum, we compute

$$\partial_\sigma^2\left(\frac{V_{\mathrm{eff}}(\sigma)}{V_{\mathrm{st}}}\right)\bigg|_{\sigma=\sigma_0} = \frac{1}{\pi} > 0\,. \tag{3.39}$$

We can now use this expression to eliminate μ_0 (and also g^2) from eq. (3.34). From now on we will set $m_0 = 1$ and express all dimensionful quantities in units of m_0:

$$\frac{V_{\mathrm{eff}}}{V_{\mathrm{st}}} = \frac{\sigma^2}{4\pi}\left(\ln\left(\sigma^2\right) - 1\right). \tag{3.40}$$

This is our full renormalized result for the effective potential.

3.2 Effective potential for $B = 0, \mu \neq 0, T \neq 0$

We now discuss the case of non-zero chemical potential and temperature. The Dirac operator reads:

$$D = \sigma + \gamma_0 \left(\partial_0 - \mu\right) + \gamma_1 \partial_1 . \tag{3.41}$$

In order to compute the effective potential, we need the eigenvalues of the squared Dirac operator $D\gamma_5 D\gamma_5$:

$$D\gamma_5 D\gamma_5 = \sigma^2 - \left(\partial_0 - \mu\right)^2 - \partial_1^2 , \tag{3.42}$$

which read

$$\begin{aligned} \lambda_p &= \sigma^2 - \left(-\mathrm{i}\omega_n - \mu\right)^2 - \left(-\mathrm{i}p_1\right)^2 & (3.43)\\ &= \sigma^2 + p_1^2 + \left(\omega_n - \mathrm{i}\mu\right)^2 , & (3.44)\end{aligned}$$

where we have introduced the Matsubara frequencies

$$\omega_n = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\right) \tag{3.45}$$

with $n \in \mathbb{Z}$ to account for a finite temperature $T = \frac{1}{\beta}$. The eigenvalues once more have a two-fold degeneracy.

Heat kernel: First we calculate the heat kernel $K(t)$ of the squared Dirac operator by using the eigenvalues λ_p (note that L is the spatial extent):

$$\begin{aligned} K(t) &= \frac{L}{\pi} \int \mathrm{d}p_1 \sum_{n\in\mathbb{Z}} \exp\left(- \left(\sigma^2 + p_1^2 + \left(\frac{\pi}{\beta} + \frac{2\pi n}{\beta} - \mathrm{i}\mu \right)^2 \right) t \right) & (3.46)\\ &= \frac{L}{\pi} \int \mathrm{d}p_1 \, \mathrm{e}^{-\left(\sigma^2+p_1^2\right)t} \frac{\beta}{2\sqrt{\pi t}} \theta_3 \left(-\frac{\mathrm{i}\beta\mu}{2} + \frac{\pi}{2}, \exp\left(-\frac{\beta^2}{4t} \right) \right) & (3.47)\\ &= \frac{V_{\mathrm{st}}}{2\sqrt{\pi^3 t}} \int \mathrm{d}p_1 \, \mathrm{e}^{-\left(\sigma^2+p_1^2\right)t} \theta_4 \left(-\frac{\mathrm{i}\beta\mu}{2}, \exp\left(-\frac{\beta^2}{4t} \right) \right) & (3.48)\\ &= \frac{V_{\mathrm{st}}}{2\sqrt{\pi^3 t}} \int \mathrm{d}p_1 \, \mathrm{e}^{-\left(\sigma^2+p_1^2\right)t} \left(1 + 2 \sum_{n=1}^{\infty} (-1)^n \mathrm{e}^{-\frac{\beta^2 n^2}{4t}} \cos\left(-\mathrm{i}\beta\mu n\right) \right) & (3.49)\\ &= \frac{V_{\mathrm{st}}}{2\sqrt{\pi^3 t}} \int \mathrm{d}p_1 \, \mathrm{e}^{-\left(\sigma^2+p_1^2\right)t} \\ &\quad + \frac{V_{\mathrm{st}}}{\sqrt{\pi^3 t}} \int \mathrm{d}p_1 \sum_{n=1}^{\infty} \mathrm{e}^{-\left(\sigma^2+p_1^2\right)t} (-1)^n \mathrm{e}^{-\frac{\beta^2 n^2}{4t}} \cosh(\beta\mu n) & (3.50)\\ &= \frac{V_{\mathrm{st}}}{2\pi} \frac{\mathrm{e}^{-\sigma^2 t}}{t} + \frac{V_{\mathrm{st}}}{\sqrt{\pi^3 t}} \int \mathrm{d}p_1 \sum_{n=1}^{\infty} (-1)^n \mathrm{e}^{-\left(\sigma^2+p_1^2\right)t - \frac{\beta^2 n^2}{4t}} \cosh(\beta\mu n) . & (3.51)\end{aligned}$$

The $\theta_3(z,q)$-function in eq. (3.48) and the $\theta_4(z,q)$-function in eq. (3.49) are the elliptic theta functions.

ζ-function renormalization: Next we calculate the associated ζ-function of the squared DIRAC operator $D\gamma_5 D\gamma_5$ [5]:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int\limits_0^\infty \mathrm{d}t\, t^{s-1} K(t) \tag{3.52}$$

$$\begin{aligned} &= \frac{V_{\mathrm{st}}}{2\pi\Gamma(s)} \int\limits_0^\infty \mathrm{d}t\, t^{s-2} \mathrm{e}^{-\sigma^2 t} \\ &\quad + \frac{V_{\mathrm{st}}}{\pi^{\frac{3}{2}}\Gamma(s)} \int \mathrm{d}p_1 \sum_{n=1}^{\infty} (-1)^n \cosh(\beta\mu n) \int\limits_0^\infty \mathrm{d}t\, t^{s-\frac{3}{2}} \mathrm{e}^{-(\sigma^2+p_1^2)t-\frac{\beta^2 n^2}{4t}} \end{aligned} \tag{3.53}$$

$$\begin{aligned} &= \frac{V_{\mathrm{st}}}{2\pi} \sigma^{2-2s} \frac{\Gamma(s-1)}{\Gamma(s)} + \frac{V_{\mathrm{st}} 2^{\frac{3}{2}}}{\pi^{\frac{3}{2}}\Gamma(s)} \int \mathrm{d}p_1 \sum_{n=1}^{\infty} (-1)^n \cosh(\beta\mu n) \\ &\quad \cdot \sqrt{\frac{\sqrt{\sigma^2+p_1^2}}{\beta n}} \left(\frac{\beta n}{2\sqrt{\sigma^2+p_1^2}} \right)^s K_{\frac{1}{2}-s}\left(\beta n \sqrt{\sigma^2+p_1^2}\right) \end{aligned} \tag{3.54}$$

$$\begin{aligned} &= \frac{V_{\mathrm{st}}}{2\pi} \frac{\sigma^{2-2s}}{s-1} + \frac{V_{\mathrm{st}} 2^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} \int \mathrm{d}p_1 \sum_{n=1}^{\infty} (-1)^n \cosh(\beta\mu n) \\ &\quad \cdot \sqrt{\frac{\sqrt{\sigma^2+p_1^2}}{\beta n}} \frac{1}{\Gamma(s)} \left(\frac{\beta n}{2\sqrt{\sigma^2+p_1^2}} \right)^s K_{\frac{1}{2}-s}\left(\beta n \sqrt{\sigma^2+p_1^2}\right) , \end{aligned} \tag{3.55}$$

where $K_\nu(z)$ is the modified BESSEL function of the second kind. In eq. (3.54) we used integral tables [5] and analytic continuation.

Now we compute $\zeta'(0)$ to obtain the effective potential, again introducing the scale μ_0 explicitly:

$$\zeta'(0) = \lim_{s\to 0} \frac{\partial}{\partial s} \zeta(s) \tag{3.56}$$

$$\begin{aligned} &= \frac{V_{\mathrm{st}}}{2\pi} \sigma^2 \left(\ln\left(\frac{\sigma^2}{\mu_0^2}\right) - 1 \right) + \frac{V_{\mathrm{st}} 2^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} \int \mathrm{d}p_1 \sum_{n=1}^{\infty} \\ &\quad \cdot \sqrt{\frac{\sqrt{\sigma^2+p_1^2}}{\beta n}} (-1)^n \cosh(\beta\mu n) \sqrt{\frac{\pi}{2}} \frac{\mathrm{e}^{-n\beta\sqrt{\sigma^2+p^2}}}{\sqrt{\beta n \sqrt{\sigma^2+p^2}}} \end{aligned} \tag{3.57}$$

$$= \frac{V_{\mathrm{st}}}{2\pi}\sigma^2\left(\ln\left(\frac{\sigma^2}{\mu_0^2}\right)-1\right)+\frac{V_{\mathrm{st}}2}{\pi\beta}\int \mathrm{d}p_1\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cosh(\beta\mu n)\mathrm{e}^{-n\beta\sqrt{\sigma^2+p^2}}\,. \tag{3.58}$$

We now use a similiar trick as in [6] to obtain

$$\begin{aligned}\zeta'(0) = {} & \frac{V_{\mathrm{st}}}{2\pi}\sigma^2\left(\ln\left(\frac{\sigma^2}{\mu_0^2}\right)-1\right)-\frac{V_{\mathrm{st}}}{\pi\beta} \\ & \cdot\int \mathrm{d}p_1\left(\ln\left(1+\mathrm{e}^{-\beta(\sqrt{\sigma^2+p_1^2}+\mu)}\right)+\ln\left(1+\mathrm{e}^{-\beta(\sqrt{\sigma^2+p_1^2}-\mu)}\right)\right)\,.\end{aligned} \tag{3.59}$$

Effective potential: We can now write down the effective potential corresponding to finite temperature and chemical potential:

$$\frac{V_{\mathrm{eff}}}{V_{\mathrm{st}}} = \frac{\sigma^2}{2g^2}+\frac{1}{2V_{\mathrm{st}}}\zeta'(0) \tag{3.60}$$

$$\begin{aligned}&= \frac{\sigma^2}{4\pi}\left(\ln\left(\sigma^2\right)-1\right)-\frac{1}{2\pi\beta} \\ &\quad\cdot\int_{-\infty}^{\infty}\mathrm{d}p_1\left[\ln\left(1+\mathrm{e}^{-\beta(\sqrt{\sigma^2+p_1^2}+\mu)}\right)+\ln\left(1+\mathrm{e}^{-\beta(\sqrt{\sigma^2+p_1^2}-\mu)}\right)\right].\end{aligned} \tag{3.61}$$

$$\begin{aligned}&= \frac{\sigma^2}{4\pi}\left(\ln\left(\sigma^2\right)-1\right)-\frac{T}{\pi} \\ &\quad\cdot\int_{0}^{\infty}\mathrm{d}p_1\left[\ln\left(1+\mathrm{e}^{-\frac{1}{T}(\sqrt{\sigma^2+p_1^2}+\mu)}\right)+\ln\left(1+\mathrm{e}^{-\frac{1}{T}(\sqrt{\sigma^2+p_1^2}-\mu)}\right)\right].\end{aligned} \tag{3.62}$$

We again have expressed all dimensionful quantities in units of m_0, see eq. (3.37). For general temperature and chemical potential the integral in eq. (3.62) is solvable only numerically. Since the logarithm terms in the integral behave for fixed μ and T like

$$\ln\left(1+a\cdot\mathrm{e}^{-b|x|}\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}a^k\cdot\mathrm{e}^{-kb|x|}=\sum_{k=1}^{\infty}\mathrm{const.}\cdot\mathrm{e}^{-kb|x|}, \tag{3.63}$$

the logarithm terms are part of the SCHWARTZ space (space of rapidly decreasing functions on $\mathbb{R}$). Therefore the integral in eq. (3.62) converges.

Finding the condensate: We know that the effective potential density is well-defined and convergent for all μ and T. In order to find the global minimum of $V_{\mathrm{eff}}/V_{\mathrm{st}}$ for given parameters we used *Mathematica*.

Fig. 3.1 shows how the effective potential density changes for different temperatures.

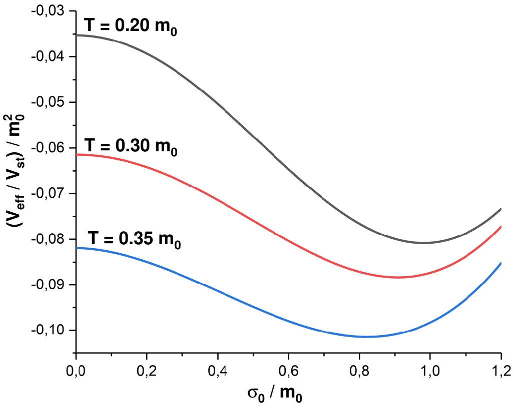

Figure 3.1: *Effective potential density for $\mu = 0.3\,m_0$ and different temperatures.*

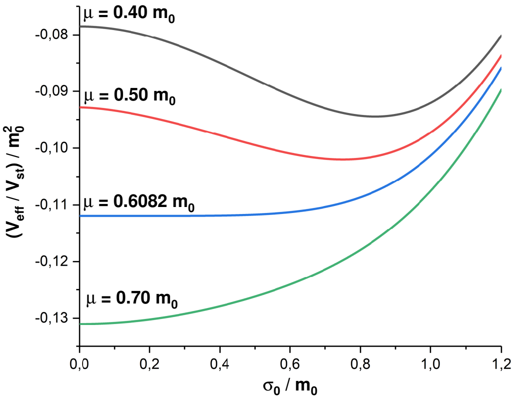

Figure 3.2: *Effective potential density for $T = 0.3183\,m_0$ and different chemical potentials. The blue line ($\mu = 0.6082\,m_0$) indicates the tricritcal point*

For larger temperatures the global minimum position of $V_{\text{eff}}/V_{\text{st}}$ shifts to smaller σ_0. The transition of the effective potential density for different chemical potentials is shown in Fig. 3.2. The change of the condensate (global minimum) between $\sigma_0 = 0$ and $\sigma_0 \neq 0$ is here well visible.

In Fig. 3.3 we show the global minimum of the effective potential as a contour plot in the μ-T-plane, and therefore obtain the μ-T-phase diagram. Furthermore, we are especially interested in the value of the critical temperature T_c for $\mu = 0$ (Fig. 3.4) and in the critical chemical potential μ_c for $T = 0$ (Fig. 3.5). We know that such a finite temperature must exist [7]. Numerically we find the following critical values:

$$T_c = 0.5669\,m_0\,, \qquad \mu_c = 0.7071\,m_0\,, \tag{3.64}$$

in accordance with the results of [8]. Further we notice that the condensate transition for $\mu = 0$ (Fig. 3.4) is continuous (second-order phase transition) and for $T = 0$ (Fig. 3.5) the condensate has a discontinuity on the critical chemical potential (first-order phase transition).

U. Wolff also noticed in [8], that there are temperatures and chemical potentials for which two local minima of the effective potential exist (coexistence region). The point that seperates the coexistence region from the region with only one minimum is called *tricritical point*. The $\mu = 0.6082\,m_0$ effective potential density in Fig. 3.2 shows this tricritical point, which reproduces the result of [8].

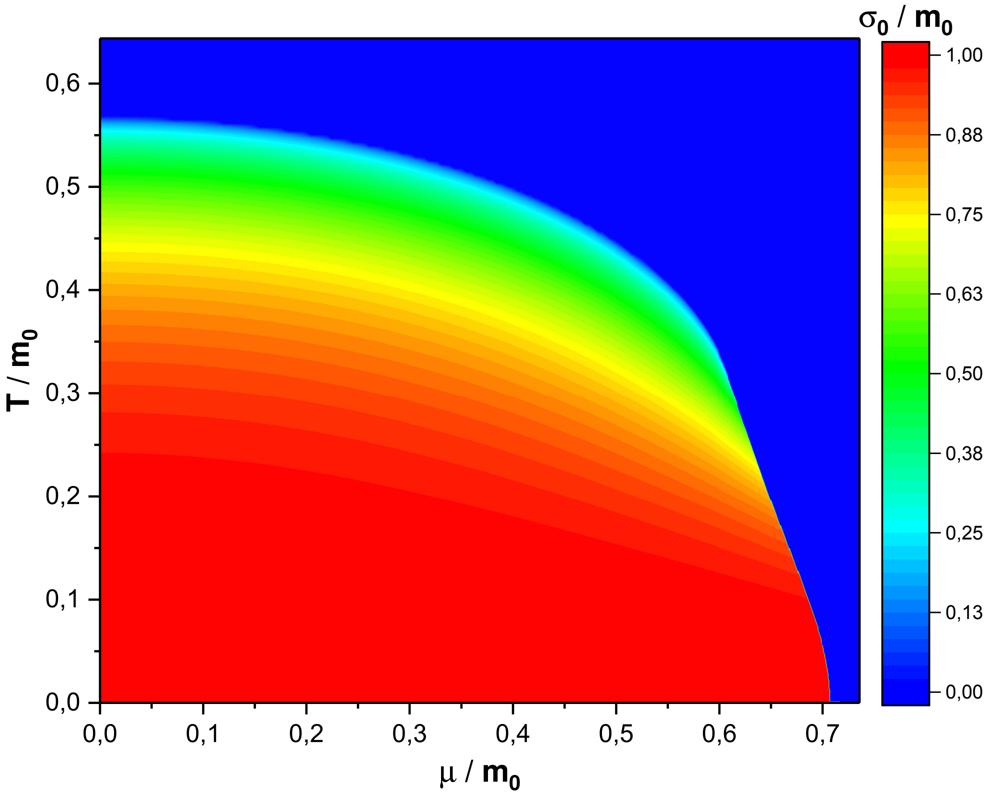

Figure 3.3: *Contour plot of the chiral condensate σ_0 in the μ-T-plane. We explicitly rescale all quantities by the value of σ_0 for $\mu = 0 = T$, m_0.*

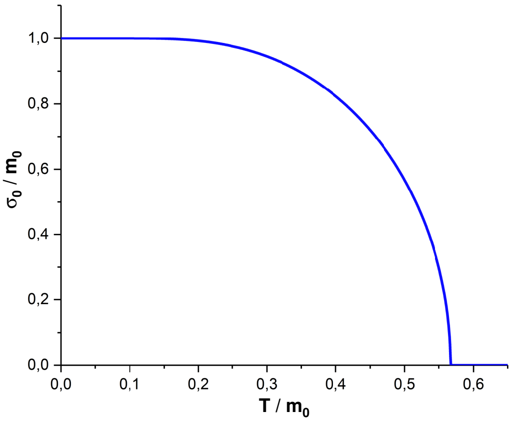

Figure 3.4: *Condensate σ_0 as a function of the temperature for $\mu = 0$.*

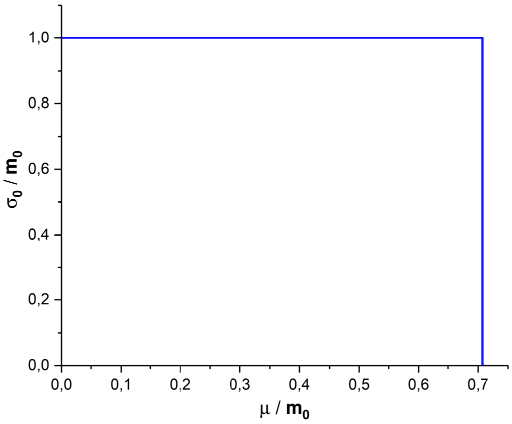

Figure 3.5: *Condensate σ_0 as a function of the chemical potential for $T = 0$.*

3.3 Effective potential for $B \neq 0, \mu \neq 0, T \neq 0$

The last case that will be discussed is the most general one, in which all exterior parameters are non-vanishing.

Calculate the eigenvalues: To compute the corresponding effective potential we again first need to find the eigenvalues of the DIRAC operator

$$D = \sigma + \gamma_0(\partial_0 - \mathrm{i}eBx_1 - \mu) + \gamma_1\partial_1. \tag{3.65}$$

To achieve this we choose the following explicit representation of gamma matrices that satisfy the anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\mathbb{1}$:

$$\gamma_0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_1 = \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{3.66}$$

where σ_1 and σ_2 are the PAULI matrices. In this representation the DIRAC operator reads

$$\begin{aligned}
D &= \sigma\mathbb{1} + \sigma_1 (\partial_0 - \mathrm{i}eBx_1 - \mu) + \sigma_2\partial_1 && (3.67)\\
&= \sigma\mathbb{1} + \begin{pmatrix} 0 & \partial_0 - \mathrm{i}eBx_1 - \mu - \mathrm{i}\partial_1 \\ \partial_0 - \mathrm{i}eBx_1 - \mu + \mathrm{i}\partial_1 & 0 \end{pmatrix} && (3.68)\\
&= \sigma\mathbb{1} - \mathrm{i} \begin{pmatrix} 0 & \mathrm{i}(\partial_0 - \mathrm{i}eBx_1) - \mathrm{i}\mu + \partial_1 \\ \mathrm{i}(\partial_0 - \mathrm{i}eBx_1) - \mathrm{i}\mu - \partial_1 & 0 \end{pmatrix} && (3.69)\\
&= \sigma\mathbb{1} - \mathrm{i} \begin{pmatrix} 0 & \omega + eBx_1 + \partial_1 - \mathrm{i}\mu \\ \omega + eBx_1 - \partial_1 - \mathrm{i}\mu & 0 \end{pmatrix} && (3.70)\\
&= \sigma\mathbb{1} - \mathrm{i} \begin{pmatrix} 0 & A - \mathrm{i}\mu \\ A^\dagger - \mathrm{i}\mu & 0 \end{pmatrix} = \sigma\mathbb{1} - \mathrm{i}\tilde{D}, && (3.71)
\end{aligned}$$

where we have introduced $A := \omega + eBx_1 + \partial_1$. We denote the eigenvalues of $\tilde{D}$ by $\tilde{\lambda}$. Since the matrix $\tilde{D}$ commutes with the identity matrix, the eigenequation of the DIRAC operator is

$$D\psi = \left(\sigma - \mathrm{i}\tilde{\lambda}\right)\psi\,. \tag{3.72}$$

Next we look for the eigenvalues $\tilde{\lambda}$ of $\tilde{D}$ by applying $\tilde{D}$ to both sides of the eigenequation

$$\tilde{D}\phi = \tilde{\lambda}\phi \quad \Rightarrow \quad \tilde{D}^2\phi = \tilde{\lambda}^2\phi\,, \tag{3.73}$$

where

$$\tilde{D}^2 = \begin{pmatrix} (A - \mathrm{i}\mu)(A^\dagger - \mathrm{i}\mu) & 0 \\ 0 & (A^\dagger - \mathrm{i}\mu)(A - \mathrm{i}\mu) \end{pmatrix} \tag{3.74}$$

$$= \begin{pmatrix} AA^\dagger - \mu^2 - \mathrm{i}\mu A - \mathrm{i}\mu A^\dagger & 0 \\ 0 & A^\dagger A - \mu^2 - \mathrm{i}\mu A - \mathrm{i}\mu A^\dagger \end{pmatrix}. \tag{3.75}$$

We continue by expanding $A^\dagger A$ and $AA^\dagger$:

$$AA^\dagger = (\omega + eBx_1 + \partial_1)(\omega + eBx_1 - \partial_1) \tag{3.76}$$

$$= (\omega + eBx_1)^2 - \partial_1^2 + \partial_1(\omega + eBx_1) - (\omega + eBx_1)\partial_1 \tag{3.77}$$

$$= (\omega + eBx_1)^2 - \partial_1^2 + eB, \tag{3.78}$$

$$A^\dagger A = (\omega + eBx_1 - \partial_1)(\omega + eBx_1 + \partial_1) \tag{3.79}$$

$$= (\omega + eBx_1)^2 - \partial_1^2 - \partial_1(\omega + eBx_1) + (\omega + eBx_1)\partial_1 \tag{3.80}$$

$$= (\omega + eBx_1)^2 - \partial_1^2 - eB. \tag{3.81}$$

We note that the difference between $A^\dagger A$ and $AA^\dagger$ is $2eB$. Both expressions can now be inserted in eq. (3.75):

$$\tilde{D}^2 = (\omega + eBx_1 - \mathrm{i}\mu)^2 \mathbb{1} - \partial_1 \mathbb{1} + eB\sigma_3. \tag{3.82}$$

It is easy to show that

$$\phi_0 = \exp\left(-\frac{1}{2eB}(\omega + eBx_1 - \mathrm{i}\mu)^2\right) \tag{3.83}$$

is a zero mode of $(A^\dagger - \mathrm{i}\mu)(A - \mathrm{i}\mu)$:

$$(A^\dagger - \mathrm{i}\mu)(A - \mathrm{i}\mu)\,\phi_0 = (A^\dagger - \mathrm{i}\mu)0 = 0 \cdot \phi_0\,. \tag{3.84}$$

Since the difference between the diagonal components of $\tilde{D}^2$ is $2eB$, we know automatically that $2eB$ must be another eigenvalue. Furthermore we can now apply $(A^\dagger - \mathrm{i}\mu)$ as a creation operator to find the remaining eigenmodes. For instance, define

$$\phi_1 = \left(A^\dagger - \mathrm{i}\mu\right)\phi_0\,, \tag{3.85}$$

to find that there exists another eigenmode with the eigenvalue $2eB$:

$$\left(A^\dagger - \mathrm{i}\mu\right)(A - \mathrm{i}\mu)\left(A^\dagger - \mathrm{i}\mu\right)\phi_0 = 2eB\left(A^\dagger - \mathrm{i}\mu\right)\phi_0 \tag{3.86}$$

$$\Rightarrow \left(A^\dagger - \mathrm{i}\mu\right)(A - \mathrm{i}\mu)\,\phi_1 = 2eB\phi_1. \tag{3.87}$$

With this concept (that we borrowed from supersymmetry) we can systematically find every eigenvalue. The ground state eigenvalue is $\tilde{\lambda}^2 = 0$ and every other eigenvalue is doubly degenerate with $\tilde{\lambda}^2 = 2eBn$ where $n \in \mathbb{N}$. Inserting the eigenvalues $\tilde{\lambda}$ in eq. (3.72) we find the eigenvalues of the DIRAC operator:

$$\lambda = \sigma \pm \mathrm{i}\sqrt{2eBn}. \tag{3.88}$$

We see that the eigenvalues come in complex conjugate pairs and therefore the determinant is real. It is noteworthy that in this theory the eigenvalues are independent of the chemical potential μ and the temperature T.

To calculate the determinant of D we use

$$\det(D) = \prod_{\lambda} \lambda_n = \sigma^k \prod_{n=1}^{\infty} \left(\sigma + \mathrm{i}\sqrt{2eBn}\right)^k \left(\sigma - \mathrm{i}\sqrt{2eBn}\right)^k \tag{3.89}$$

$$= \sigma^k \prod_{n=1}^{\infty} \left(\sigma^2 + 2eBn\right)^k \tag{3.90}$$

where $k = \frac{V_{\mathrm{st}} eB}{2\pi}$ is a degeneracy factor [6]. Thus,

$$\det(D) = \sigma^k \prod_{n=1}^{\infty} \left(\sigma^2 + 2eBn\right)^k , \tag{3.91}$$

because the eigenvalues come in complex conjugate pairs. Next we separate the ground state and the degenerate excited states and find

$$\ln\left(\det(D)\right) = k \ln\left(\frac{\sigma}{\mu_0}\right) + k \sum_{n=1}^{\infty} \ln\left(\frac{\sigma^2 + 2eB}{\mu_0^2}\right) \tag{3.92}$$

$$= \frac{k}{2} \ln\left(\frac{\sigma^2}{\mu_0^2}\right) + k \sum_{n=1}^{\infty} \ln\left(\frac{\sigma^2 + 2eB}{\mu_0^2}\right) . \tag{3.93}$$

The effective potential density reads

$$\frac{V_{\mathrm{eff}}}{V_{\mathrm{st}}} = \frac{\sigma^2}{2g^2} - \frac{1}{V_{\mathrm{st}}} \ln\left(\det\left(D\right)\right) . \tag{3.94}$$

We calculate the (divergent) sum using the ζ-function renormalization prescription:

$$\sum_{n=1}^{\infty} \ln\left(\frac{\sigma^2 + 2eBn}{\mu_0^2}\right) = -\zeta'(0) \tag{3.95}$$

where we define the ζ-function as

$$\zeta(s) = \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int\limits_0^{\infty} \mathrm{d}t\, t^{s-1} \mathrm{e}^{-\frac{\sigma^2+2eBn}{\mu_0^2} t} \,. \tag{3.96}$$

Taking the t-integral and then the s-derivative immediately proves eq. (3.95).

We proceed to calculate the ζ-function by taking the sum first:

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)} \int\limits_0^{\infty} \mathrm{d}t\, t^{s-1} \mathrm{e}^{-\frac{\sigma^2}{\mu_0^2}t - \frac{2eBn}{\mu_0^2}t} && (3.97)\\
&= \frac{1}{\Gamma(s)} \left(\frac{\mu_0^2}{2eB}\right)^s \Gamma(s) \zeta\left(s, \left(\sigma^2 + 2eB\right) \frac{1}{\mu_0^2} \frac{\mu_0^2}{2eB}\right) && (3.98)\\
&= \left(\frac{\mu_0^2}{2eB}\right)^s \zeta\left(s, 1 + \frac{\sigma^2}{2eB}\right) \,, && (3.99)
\end{aligned}$$

where $\zeta(s, a)$ is the Hurwitz ζ-function. Thus

$$\begin{aligned}
\zeta'(0) &= \ln\left(\frac{\mu_0^2}{2eB}\right) \zeta\left(0, 1 + \frac{\sigma^2}{2eB}\right) + \ln\left(\Gamma\left(1 + \frac{\sigma^2}{2eB}\right)\right) - \frac{1}{2}\ln(2\pi) && (3.100)\\
&= -\ln\left(\frac{2eB}{\mu_0^2}\right)\left(\frac{1}{2} - 1 - \frac{\sigma^2}{2eB}\right) + \ln\left(\frac{\sigma^2}{2eB}\Gamma\left(\frac{\sigma^2}{2eB}\right)\right) - \frac{1}{2}\ln(2\pi) && (3.101)\\
&= \ln\left(\frac{2eB}{\mu_0^2}\right)\left(\frac{1}{2} + \frac{\sigma^2}{2eB}\right) + \ln\left(\frac{\sigma^2}{2eB}\right) + \ln\left(\Gamma\left(\frac{\sigma^2}{2eB}\right)\right) - \frac{\ln(2\pi)}{2} \,. && (3.102)
\end{aligned}$$

Next we use eq. (3.92) to get:

$$\begin{aligned}
\ln\left(\det(D)\right) = \frac{k}{2}\ln\left(\frac{\sigma^2}{\mu_0^2}\right) &- k\ln\left(\frac{2eB}{\mu_0^2}\right)\left(\frac{1}{2} + \frac{\sigma^2}{2eB}\right)\\
&- k\ln\left(\frac{\sigma^2}{2eB}\right) - k\ln\left(\Gamma\left(\frac{\sigma^2}{2eB}\right)\right) + \frac{k}{2}\ln(2\pi) \,,
\end{aligned} \tag{3.103}$$

and, from eq. (3.94):

$$\begin{aligned}\frac{V_{\text{eff}}}{V_{\text{st}}} &= \frac{\sigma^2}{2g^2} - \frac{k}{2V_{\text{st}}}\ln\left(\frac{\sigma^2}{\mu_0^2}\right) + \frac{k}{V_{\text{st}}}\ln\left(\frac{2eB}{\mu_0^2}\right)\left(\frac{1}{2} + \frac{\sigma^2}{2eB}\right) \\ &\quad + \frac{k}{V_{\text{st}}}\ln\left(\frac{\sigma^2}{2eB}\right) + \frac{k}{V_{\text{st}}}\ln\left(\Gamma\left(\frac{\sigma^2}{2eB}\right)\right) - \frac{k}{2V_{\text{st}}}\ln(2\pi)\end{aligned} \tag{3.104}$$

$$\begin{aligned}&= \frac{\sigma^2}{2g^2} - \frac{eB}{4\pi}\ln\left(\frac{\sigma^2}{\mu_0^2}\right) + \frac{eB}{2\pi}\ln\left(\frac{2eB}{\mu_0^2}\right)\left(\frac{1}{2} + \frac{\sigma^2}{2eB}\right) \\ &\quad + \frac{eB}{2\pi}\ln\left(\frac{\sigma^2}{2eB}\right) + \frac{eB}{2\pi}\ln\left(\Gamma\left(\frac{\sigma^2}{2eB}\right)\right) - \frac{eB}{4\pi}\ln(2\pi)\,.\end{aligned} \tag{3.105}$$

Now we use $\mu_0^2 = m_0^2 \mathrm{e}^{\frac{2\pi}{g^2}}$:

$$\begin{aligned}\frac{V_{\text{eff}}}{V_{\text{st}}} &= \frac{\sigma^2}{2g^2} - \frac{eB}{4\pi}\left[\ln\left(\frac{\sigma^2}{m_0^2}\right) - \frac{2\pi}{g^2}\right] + \frac{eB}{2\pi}\left[\ln\left(\frac{2eB}{m_0^2} - \frac{2\pi}{g^2}\right)\right]\left(\frac{1}{2} + \frac{\sigma^2}{2eB}\right) \\ &\quad + \frac{eB}{2\pi}\ln\left(\frac{\sigma^2}{2eB}\right) + \frac{eB}{2\pi}\ln\left[\Gamma\left(\frac{\sigma^2}{2eB}\right)\right] - \frac{eB}{4\pi}\ln(2\pi)\end{aligned} \tag{3.106}$$

$$\begin{aligned}&= \frac{\sigma^2}{2g^2} + \frac{eB}{2g^2} - \frac{eB}{g^2}\left(\frac{1}{2} + \frac{\sigma^2}{2eB}\right) \\ &\quad - \frac{eB}{4\pi}\ln\left(\frac{\sigma^2}{m_0^2}\right) + \frac{eB}{2\pi}\ln\left(\frac{2eB}{m_0^2}\right)\left(\frac{1}{2} + \frac{\sigma^2}{2eB}\right) \\ &\quad + \frac{eB}{2\pi}\ln\left(\frac{\sigma^2}{2eB}\right) + \frac{eB}{2\pi}\ln\left[\Gamma\left(\frac{\sigma^2}{2eB}\right)\right] - \frac{eB}{4\pi}\ln(2\pi)\end{aligned} \tag{3.107}$$

$$\begin{aligned}&= -\frac{eB}{4\pi}\ln\left(\frac{2\pi\sigma^2}{m_0^2}\right) + \frac{eB}{2\pi}\ln\left(\frac{2eB}{m_0^2}\right)\left(\frac{1}{2} + \frac{\sigma^2}{2eB}\right) \\ &\quad + \frac{eB}{2\pi}\ln\left(\frac{\sigma^2}{2eB}\right) + \frac{eB}{2\pi}\ln\left[\Gamma\left(\frac{\sigma^2}{2eB}\right)\right]\,.\end{aligned} \tag{3.108}$$

Now we can set $m_0 = 1$ and express everything in units of m_0:

$$\begin{aligned}\frac{V_{\text{eff}}}{V_{\text{st}}} &= -\frac{eB}{4\pi}\ln\left(2\pi\sigma^2\right) + \frac{eB}{2\pi}\ln\left(2eB\right)\left(\frac{1}{2} + \frac{\sigma^2}{2eB}\right) \\ &\quad + \frac{eB}{2\pi}\ln\left(\frac{\sigma^2}{2eB}\right) + \frac{eB}{2\pi}\ln\left[\Gamma\left(\frac{\sigma^2}{2eB}\right)\right]\,.\end{aligned} \tag{3.109}$$

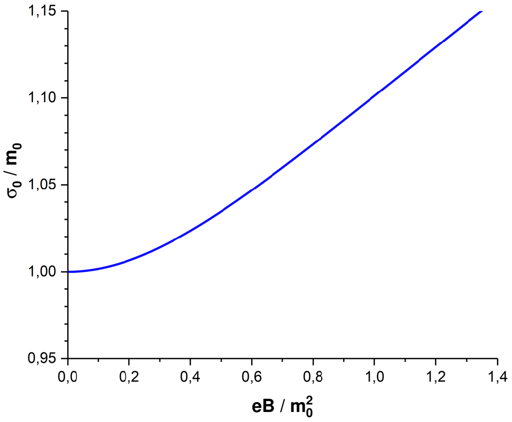

Figure 3.6: *Condensate σ_0 as a function of the magnetic field strength for $eB \to 0$.*

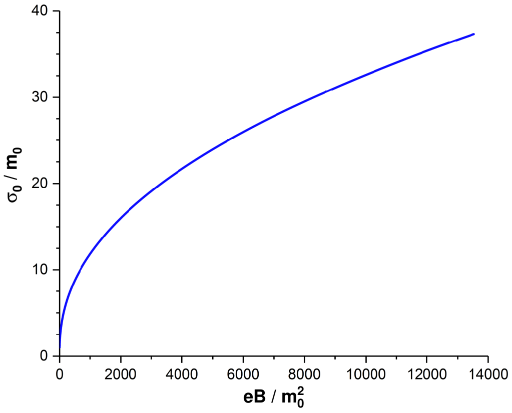

Figure 3.7: *Condensate σ_0 as a function of the magnetic field strength.*

Lastly we plot the potential density minimum over the magnetic field. We find that for $eB \to 0$ the condensate goes to $\sigma_0 = 1$ (cf. Fig. 3.6), as it should, and rises for $eB \to \infty$ to $\sigma_0 \to \infty$ (cf. Fig. 3.7).

We observe that a finite magnetic field enhances chiral symmetry breaking, an effect known in the literature as magnetic catalysis [6, 9]. In particular, there is no chiral symmetry restoration at large B, as would be the case for large μ or T (see the previous subsection).

4 Discussion

In this last chapter we summarize the main results of this thesis. In the simplest case of the Dirac operator in which all external parameters vanish we found that the condensate does not vanish and therefore the chiral symmetry is spontaneously broken.

Next we confirmed the result of [8] for the critical temperature T_c and the critical chemical potential μ_c. Beyond that we also found that for $T = 0$ the condensate has a discontinuity (first-order phase transition) in the chemical potential, while otherwise, for $\mu = 0$, the condensate as a function of T is continuous (second-order phase transition).
We also confirmed the tricritical point of [8] between the first- and second-order phase transition regions.

In the last case none of the external parameters vanished. We find (by using methods of supersymmetry) that for arbitrarily small external magnetic fields the theory becomes completely independent of temperature and chemical potential. For all magnetic field strengths the chiral symmetry is broken and is indeed enhanced for stronger magnetic fields. This is known as the phenomenon of magnetic catalysis.

Bibliography

[1] David J. Gross and André Neveu. *Dynamical symmetry breaking in asymptotically free field theories.* 1974.

[2] Alan Chodos and Hisakazu Minakata. *The Gross-Neveu model as an effective theory for polyacetylene.* 1994.

[3] V. P. Gusynin, V. A. Miransky and I. A. Shovkovy. *Catalysis of Dynamical Flavor Symmetry Breaking by Magnetic Field in* $2+1$ *Dimensions.* 1994.

[4] S. W. Hawking. *Zeta Function Regularization of Path Integrals in Curved Spacetime.* 1977.

[5] Harry Bateman. *Table of Integral Transforms.* California Institute of Technology, 1954.

[6] V. P. Gusynin, V. A. Miransky and I. A. Shovkovy. *Dynamical flavor symmetry breaking by a magnetic field in* $2+1$ *dimensions.* 1995.

[7] Roger F. Dashen, Shang-keng Ma and R. Rajaraman. *Finite-temperature behavior of a relativistic field theory with dynamical symmetry breaking.* 1975.

[8] Ulli Wolff. *The phase diagram of the infinite-N Gross-Neveu model at finite temperature and chemical potential.* 1985.

[9] K. G. Klimenko. *Three-dimensional Gross-Neveu model in an external magnetic field.* 1991.

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Selbstständigkeit und Veröffentlichung

Ich erkläre, die vorliegende Arbeit selbstständig verfasst, und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet zu haben.

Jena, den 29. September 2020

Marcel Horstmann